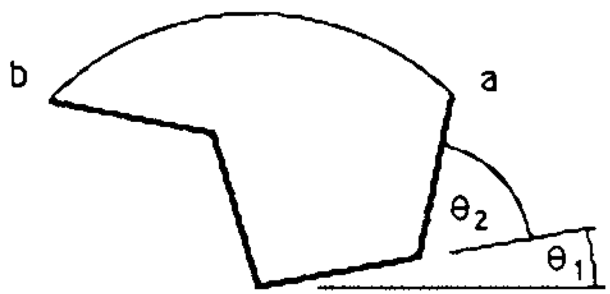
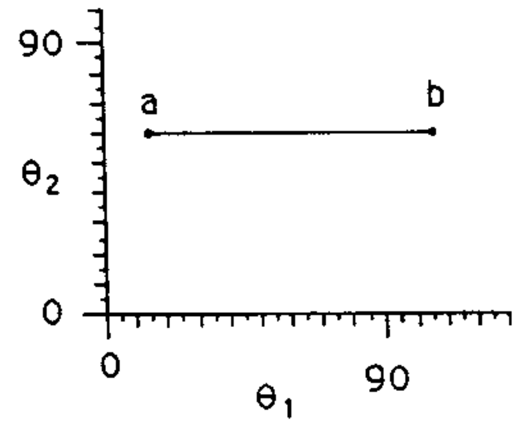
Trajectory generation and feedback control

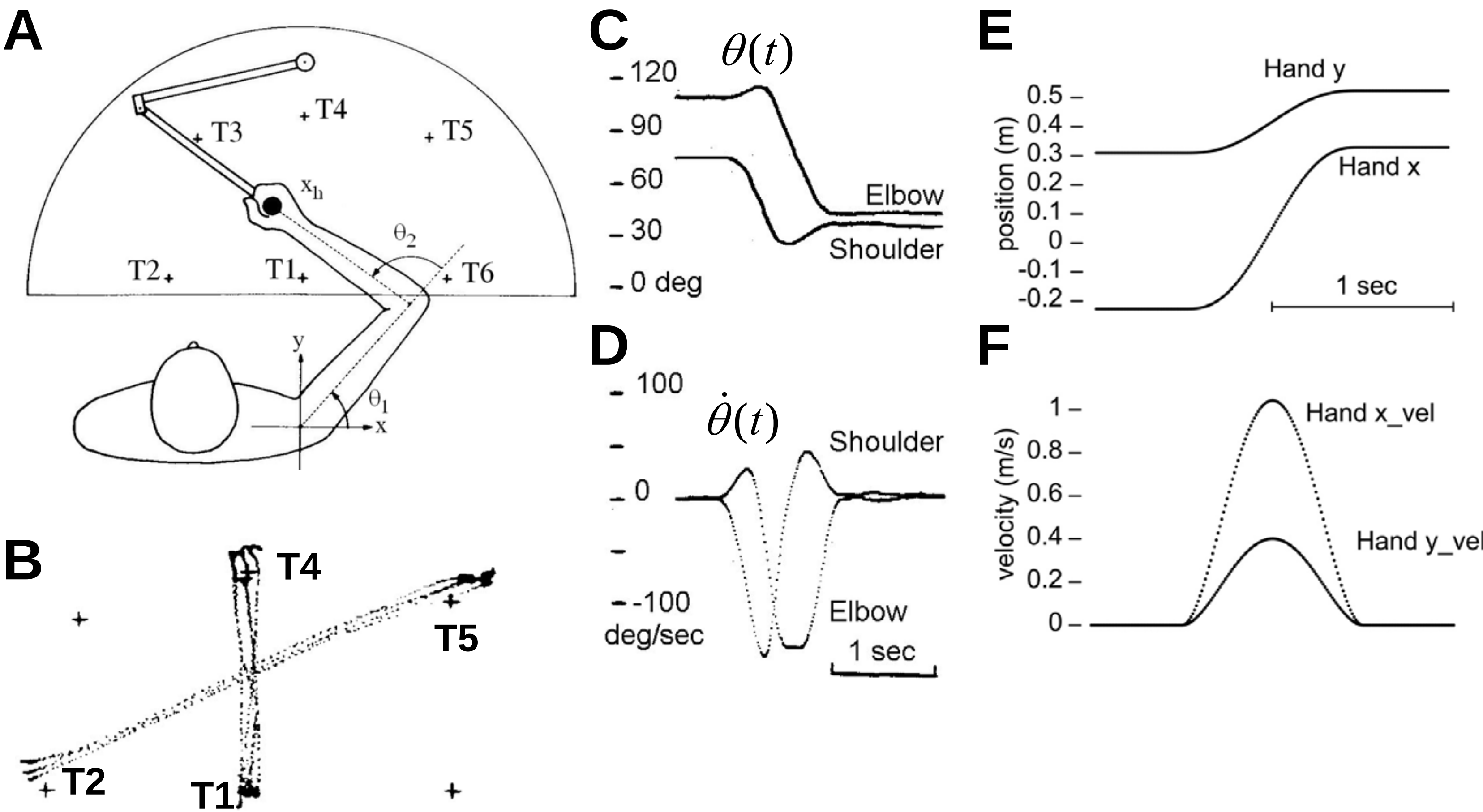
A reaching movement can entail an infinite number of trajectories from the end effector's starting location to the target.

For most reaching movements, your CNS plans the movement so that the end effector moves along just one of these trajectories: an approximately straight path with a smooth, unimodal velocity profile.

Given the choice between a trajectory that looks straight in visual coordinates and one that is straight in reality, your CNS generates a visually straight trajectory.

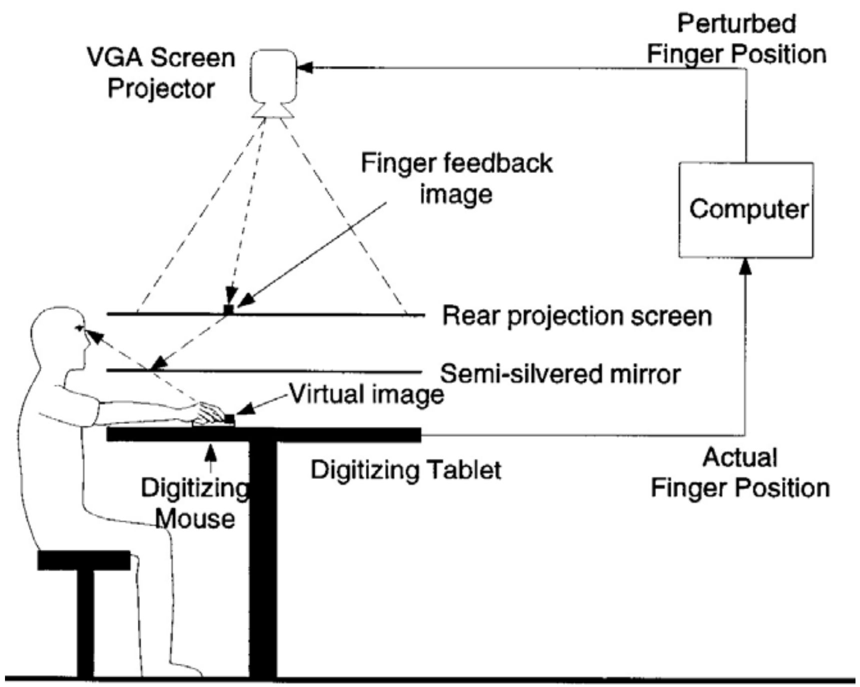

b
a
θ2
θ1

90
a
b
θ2
0
0
90
θ1

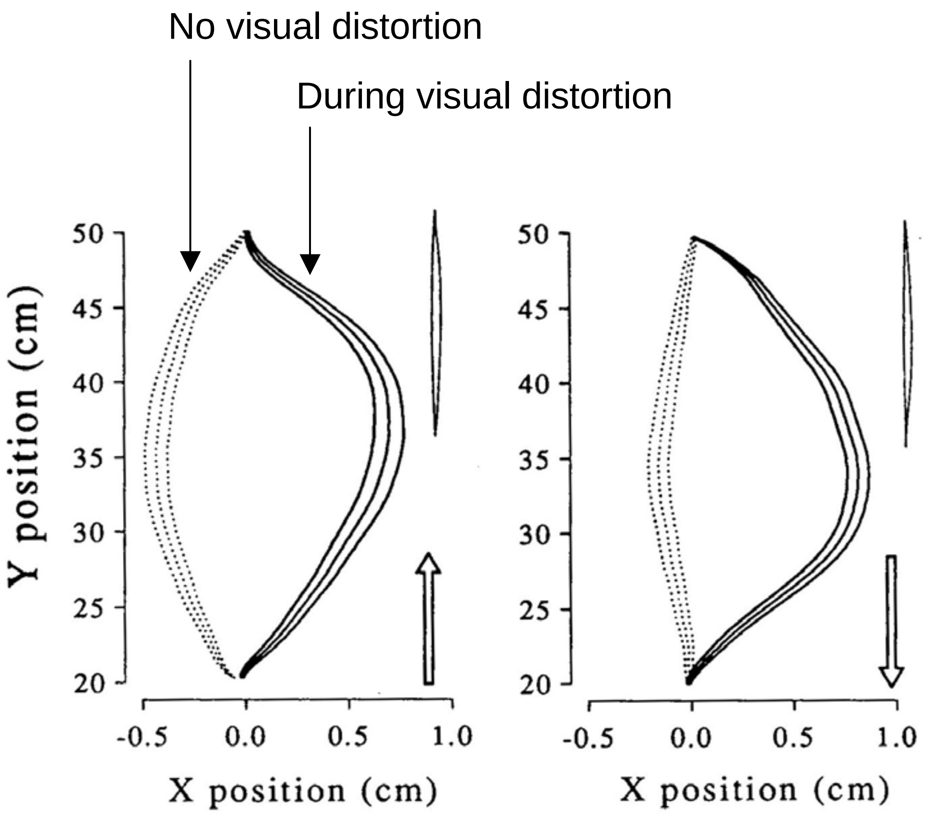
T2 -> T5



Morasso 1981

Distorting visual feedback during a reach

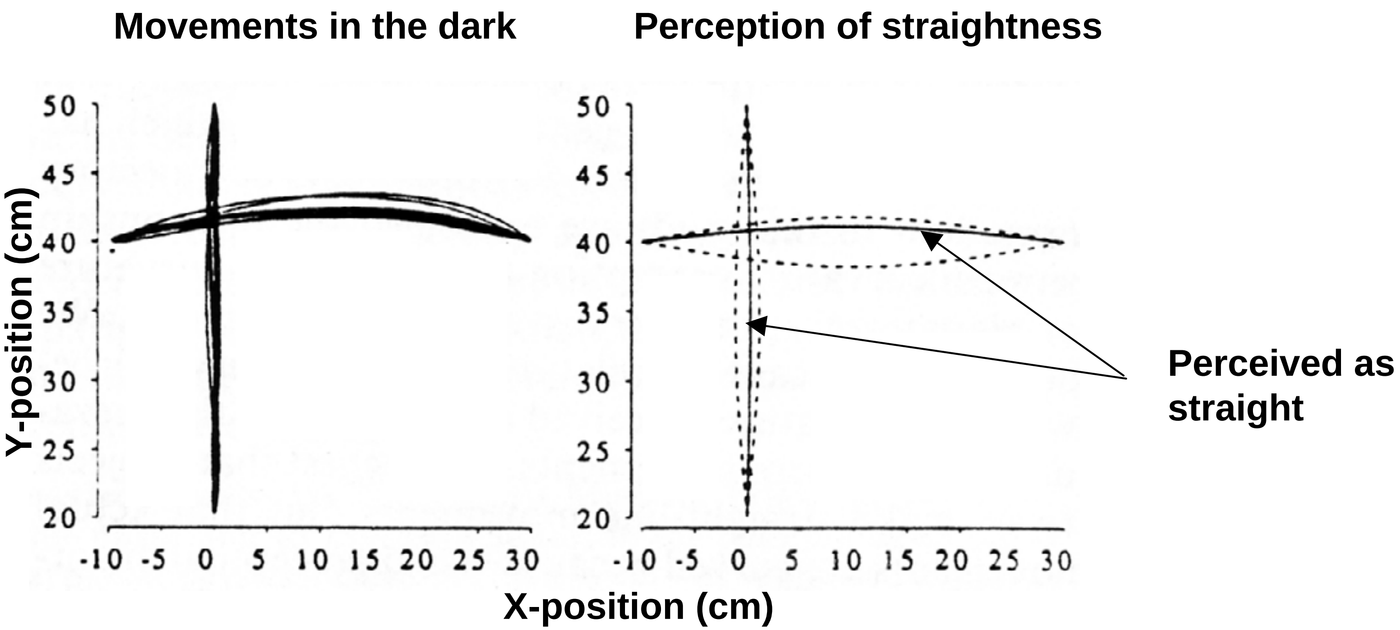


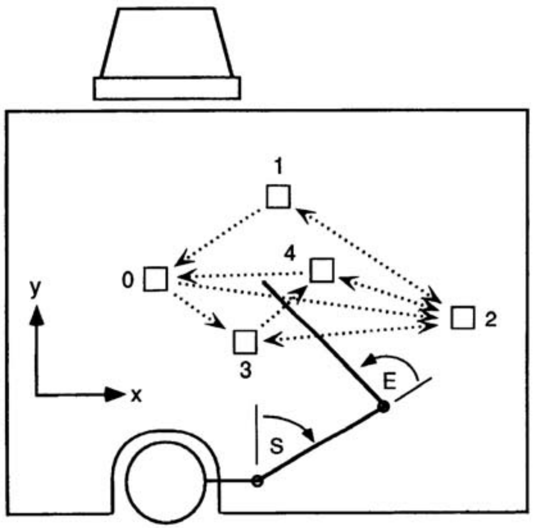


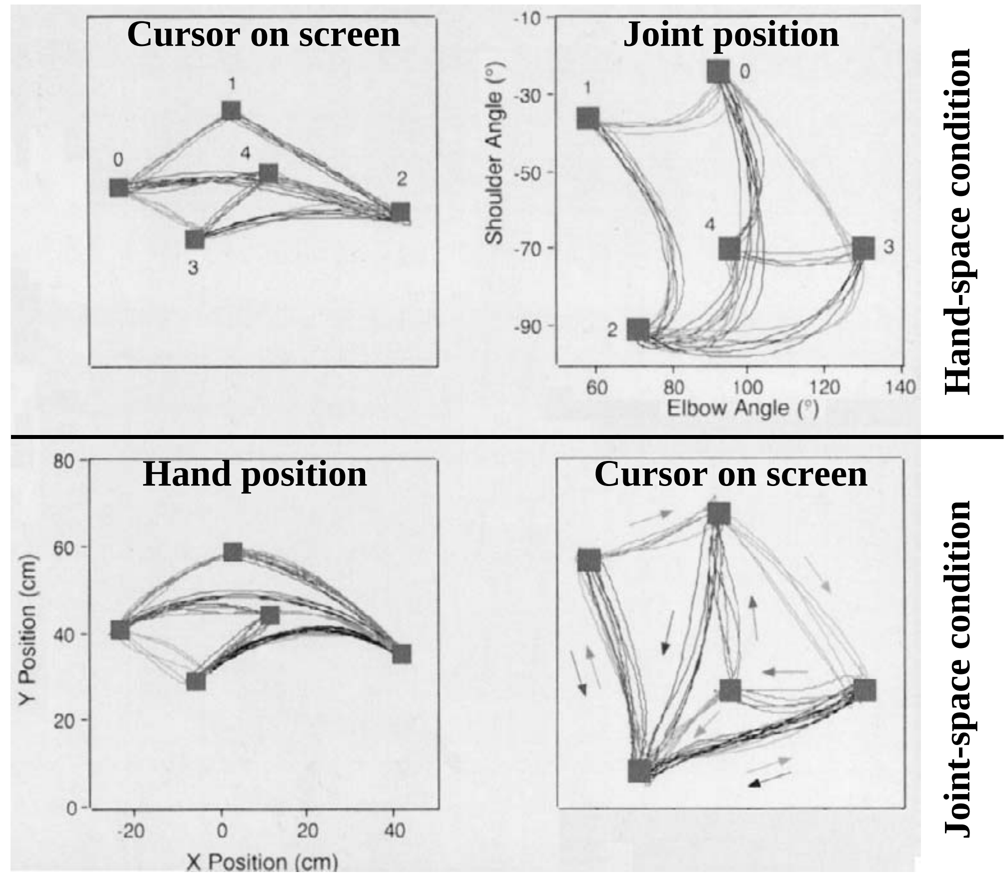
Displayed hand position = actual hand position - sinusoid
Sinusoid distorted hand position to the left.

Wolpert et al. 1995

Slight curvature in movements may be due to distortion in visual perception

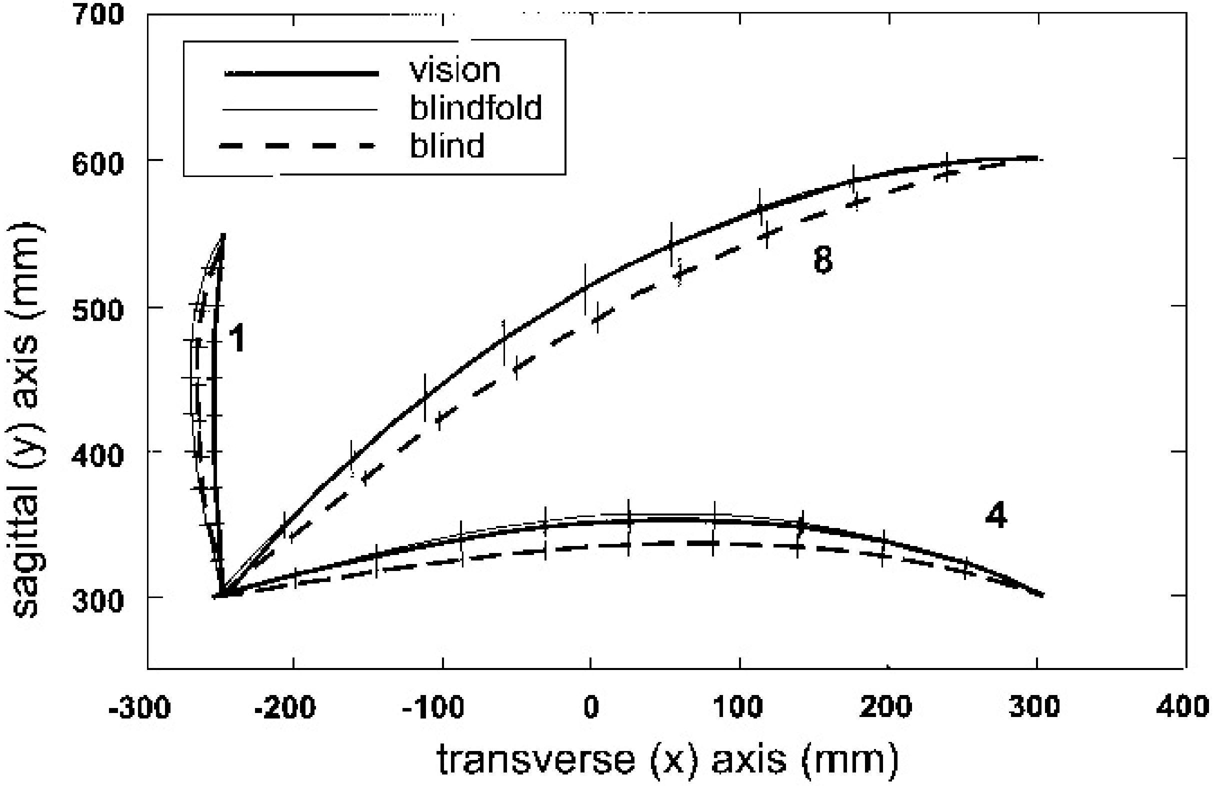






Flanagan and Rao (1995)

Reaching movements in the visually impaired



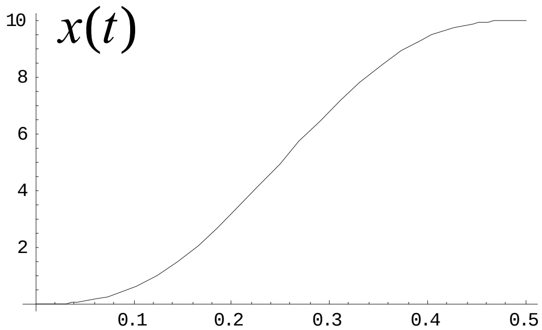
Sergio and Scott 1998

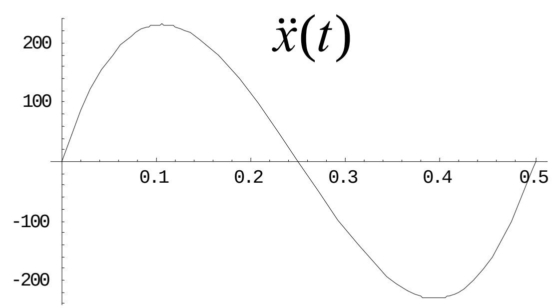
Computing a maximally smooth trajectory for point to point movements

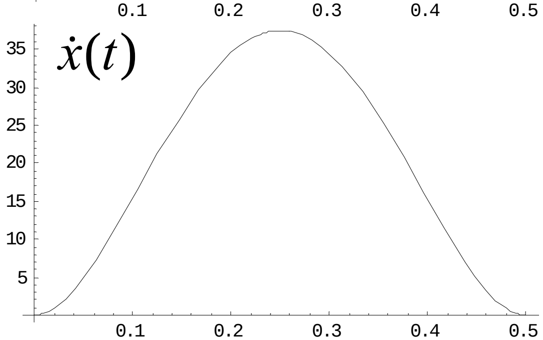
Smoothness can be quantified as the mean squared magnitude of jerk. Jerk is the rate of change of acceleration with respect to time, hence the third time derivative of position.

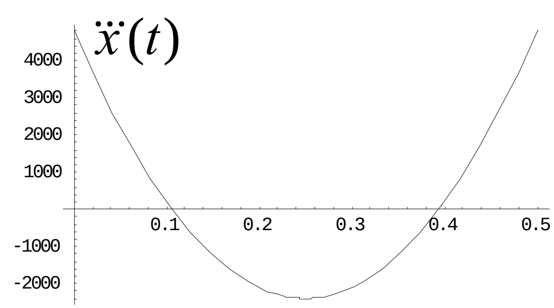
$$\text{jerk:} \quad \dddot{x}(t) = \frac{d^3x}{dt^3}$$

$$\text{smoothness cost function:} \quad \frac{1}{2}\int_{t=0}^{\text{end}} \dddot{x}^2 \, dt$$



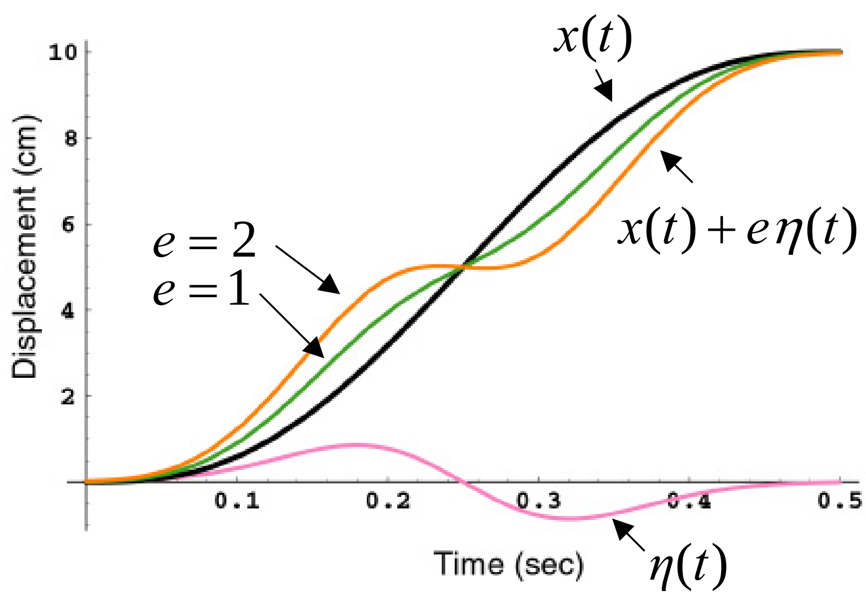






Objective: find the function $x(t)$ that minimizes the cost $H(x(t)) = \frac{1}{2}\int_{t=0}^{0.5} \ddot{x}^2 dt$

Approach: calculus of variations



$$x(t) \mapsto x(t) + e\eta(t)$$

$$\left.\frac{dH(x+e\eta)}{e}\right|_{e=0} = 0$$

$$\text{Properties of } \eta(t): \begin{cases} \eta(0) = 0 & \eta(0.5) = 0 \\ \dot{\eta}(0) = 0 & \dot{\eta}(0.5) = 0 \\ \ddot{\eta}(0) = 0 & \ddot{\eta}(0.5) = 0 \end{cases}$$

Minimizing the cost function: calculus of variation

$$H(x(t)) = \frac{1}{2}\int_0^{0.5} \ddot{x}(t)^2\, dt$$

$$x(t) \mapsto x(t) + e\eta(t)$$

$$H(x+e\eta) = \frac{1}{2}\int_0^{0.5} (\ddot{x} + e\ddot{\eta})^2\, dt$$

$$\frac{dH(x+e\eta)}{e} = \int_0^{0.5} (\ddot{x} + e\ddot{\eta})\ddot{\eta}\; dt$$

$$\left.\frac{dH(x+e\eta)}{e}\right|_{e=0} = \int_0^{0.5} \ddot{x}\,\ddot{\eta}\; dt$$

$$\int_0^{0.5} \ddot{x}\,\dddot{\eta}\; dt = \int_0^{0.5} u\; dv = uv\Big|_0^{0.5} - \int_0^{0.5} v\; du$$

$$u = \ddot{x},\; dv = \dddot{\eta}\; dt,\; du = x^{(4)}dt,\; v = \ddot{\eta}$$

$$\int_0^{0.5} \ddot{x}\,\dddot{\eta}\; dt = \ddot{x}\,\ddot{\eta}\Big|_0^{0.5} - \int_0^{0.5} \ddot{\eta}\; x^{(4)}dt = -\int_0^{0.5} \ddot{\eta}\; x^{(4)}dt$$

$$-\int_0^{0.5} \ddot{\eta}\; x^{(4)}dt = -\int_0^{0.5} u\;\; dv = -uv\Big|_0^{0.5} + \int_0^{0.5} v\; du$$

$$u = x^{(4)},\quad dv = \ddot{\eta}\; dt,\quad du = x^{(5)}dt,\quad v = \dot{\eta}$$

$$-\int_0^{0.5} \ddot{\eta}\; x^{(4)}dt = -x^{(4)}\dot{\eta}\Big|_0^{0.5} + \int_0^{0.5} \dot{\eta}\; x^{(5)}dt = \int_0^{0.5} \dot{\eta}\; x^{(5)}dt$$

$$\int_0^{0.5} \dot{\eta}\; x^{(5)}dt = x^{(5)}\eta\Big|_0^{0.5} - \int_0^{0.5} \eta\; x^{(6)}dt = -\int_0^{0.5} \eta\; x^{(6)}dt$$

$$\left.\frac{dH(x+e\eta)}{e}\right|_{e=0} = \int_0^{0.5} \eta x^{(6)} dt \equiv 0$$

Above must hold true for any function $\eta(t)$. Therefore : $x^{(6)} = 0$

$$x^{(6)} = 0 \text{ has general solution of : } x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

$$\to \dot{x}(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4$$

$$\to \ddot{x}(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3$$

Initial conditions

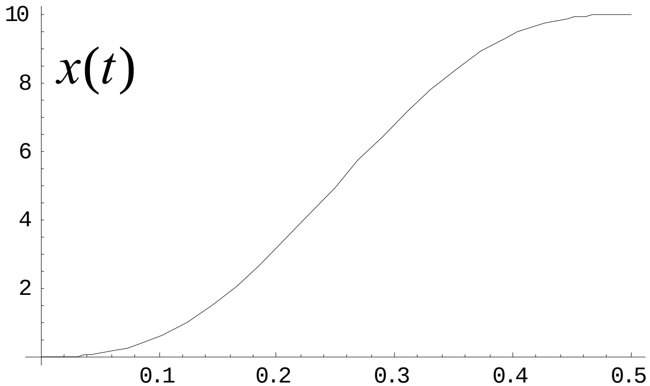
$$\begin{cases} x(0) = 0 \to a_0 = 0, \quad x(0.5) = 10 \to a_3(0.5)^3 + a_4(0.5)^4 + a_5(0.5)^5 = 10 \\ \dot{x}(0) = 0 \to a_1 = 0, \quad \dot{x}(0.5) = 0 \to 3a_3(0.5)^2 + 4a_4(0.5)^3 + 5a_5(0.5)^4 = 0 \\ \ddot{x}(0) = 0 \to a_2 = 0, \quad \ddot{x}(0.5) = 0 \to 6a_3(0.5) + 12a_4(0.5)^2 + 20a_5(0.5)^3 = 0 \end{cases}$$

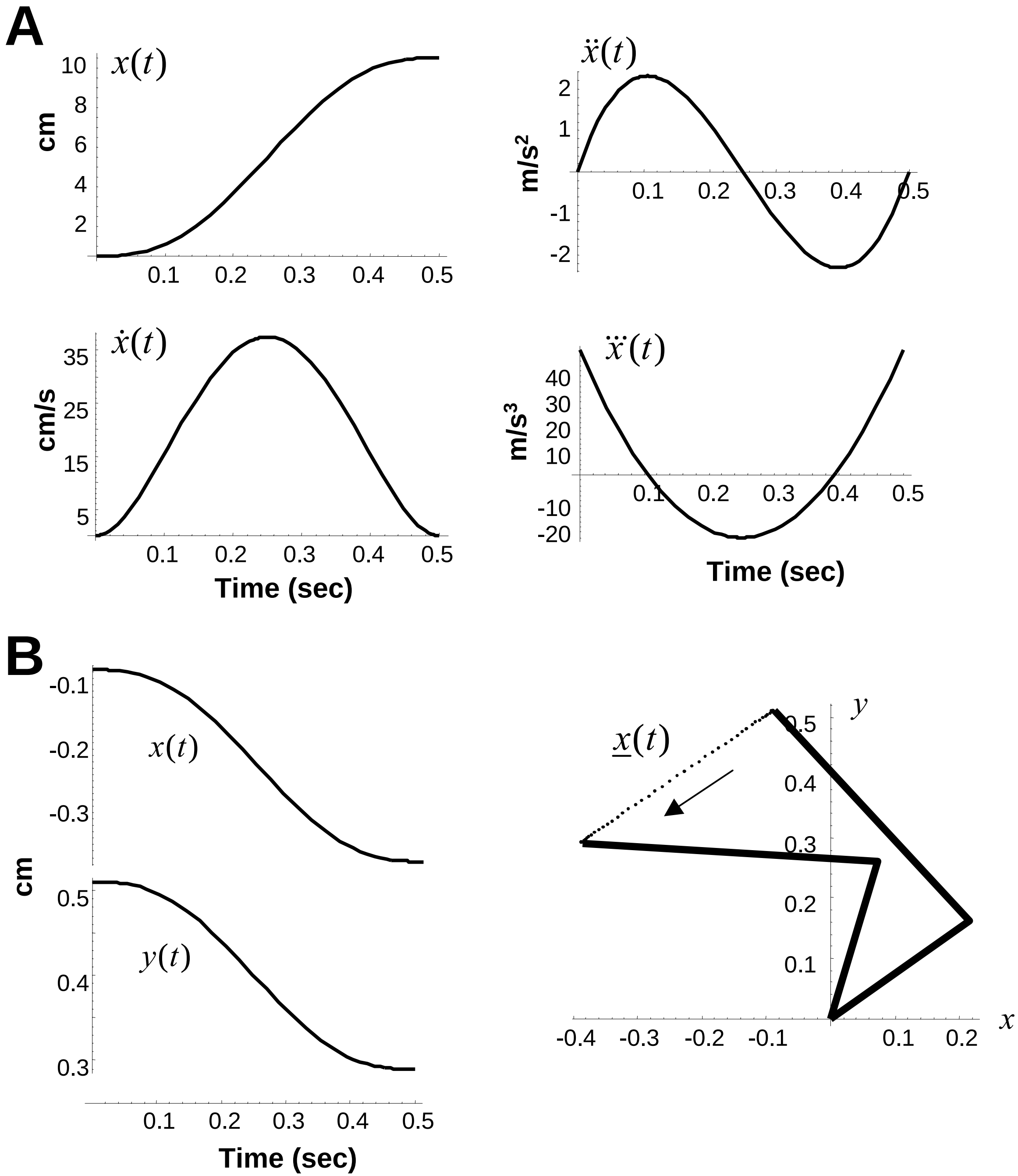
$$a_3 = 800, a_4 = -2400, a_5 = 1920$$

$$\Rightarrow \boxed{x(t) = 800t^3 - 2400t^4 + 1920t^5} \quad 0 \le t \le 0.5 \text{sec}$$

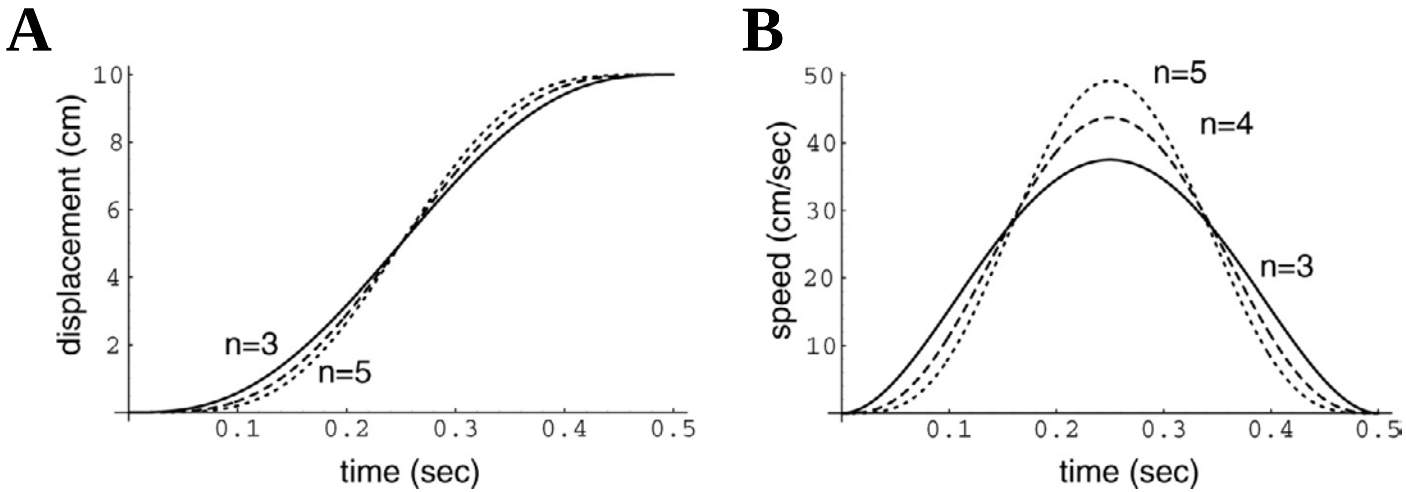
$$\text{This function minimizes the cost : } H(x(t)) = \frac{1}{2}\int_0^{0.5} \dddot{x}^2 dt$$

This is called the minimum jerk trajectory in one dimension





A
$x(t)$
cm
10
8
6
4
2
0.1
0.2
0.3
0.4
0.5
$\ddot{x}(t)$
m/s²
2
1
-1
-2
$\dot{x}(t)$
cm/s
35
25
15
5
Time (sec)
$\dddot{x}(t)$
m/s³
40
30
20
10
-10
-20
Time (sec)
B
-0.1
-0.2
-0.3
$x(t)$
cm
0.5
0.4
0.3
$y(t)$
Time (sec)
$\underline{x}(t)$
y
x
-0.4
-0.3
-0.2
-0.1
0.1
0.2


A
displacement (cm)
10
8
6
4
2
n=3
n=5
0.1
0.2
0.3
0.4
0.5
time (sec)
B
speed (cm/sec)
50
40
30
20
10
n=5
n=4
n=3
0.1
0.2
0.3
0.4
0.5
time (sec)