What is dynamics?
A description of how forces acting on a system result in motion of that system.

Example: A ball of mass $m$ is held 20 m off the ground. The force acting on the ball is the force of gravity: $f=-mg$ where $g=9.8 \text{ m/s}^2$. If we drop the ball, its dynamics are describe by:

\[
x(0) = 20 \text{ m} \\
x(0) = 0 \text{ m/s} \\
f = -mg \\
\dot{x}(t) = \frac{f}{m} = -g = -9.8 \quad \rightarrow \quad x(t) = 20 - 4.9t^2
\]

Example: Dynamics of a single joint system with mass $m$ and length $l$.

\[
\ddot{q} = \frac{1}{ml^2}(\tau - mgl \cos q)
\]
Path of motion of a system is one that minimizes an energy cost

Imagine a point mass that is at position $x_1$ at time $t_1$ and ends up at position $x_2$ at $t_2$, for example: a ball falling from a height. The trajectory that it follows to get to $x_2$ is only one of an infinite number of pathways that it could have followed. But the point mass will always follow that same trajectory $x(t)$, given the same initial conditions. What is so special about the trajectory $x(t)$ that it actually does follow?

The trajectory $x(t)$ minimizes the following cost function:

$$H(x(t)) = \int_{t_1}^{t_2} (KE - PE) \, dt = \int_{t_1}^{t_2} L(x, \dot{x}, t) \, dt$$

For a point mass:

$$\begin{cases} 
KE = \frac{1}{2} m \dot{x}^2 \\
PE = mgx 
\end{cases}$$

\[
\dot{x}(t) = -9.8 \quad \rightarrow \quad x(t) = 20 - 4.9t^2
\]

\[H(x(t)) = \int_0^{2.02} \left( \frac{1}{2} m \dot{x}(t)^2 - mgx(t) \right) \, dt = -264
\]

\[
x(t) = 20 + 20(-1.21t^3 + 0.9t^4 - 0.178t^5)
\]

\[H(x(t)) = \int_0^{2.02} \left( \frac{1}{2} m \dot{x}(t)^2 - mgx(t) \right) \, dt = -113
\]
Solving the functional for a point mass

\[ H(x(t)) = \int_{t_1}^{t_2} L \, dt = \int_{t_1}^{t_2} (KE - PE) \, dt = \int_{t_1}^{t_2} \frac{1}{2} m\dot{x}^2 - mgx \, dt \]

\[ x(t) \rightarrow x(t) + e\eta(t) \]

\[ H(x + e\eta) = \int_{t_1}^{t_2} \frac{1}{2} m(\dot{x} + e\dot{\eta})^2 \, dt - \int_{t_1}^{t_2} mg(x + e\eta) \, dt \]

\[ \frac{dH}{de} \bigg|_{e=0} = \int_{t_1}^{t_2} m(\dot{x} + e\dot{\eta})\dot{\eta} \, dt - \int_{t_1}^{t_2} mg\eta \, dt \]

\[ \frac{dH}{de} \bigg|_{e=0} = \int_{t_1}^{t_2} m\dot{x}\dot{\eta} \, dt - \int_{t_1}^{t_2} mg\eta \, dt \]

\[ u = \dot{x} \quad dv = \dot{\eta} \, dt \quad du = \ddot{x} \, dt \quad v = \eta \]

\[ \int_{t_1}^{t_2} m\dot{x}\dot{\eta} \, dt = m\dot{x}\eta|_{t_1}^{t_2} - \int_{t_1}^{t_2} m\eta\ddot{x} \, dt = -\int_{t_1}^{t_2} m\ddot{x}\eta \, dt \]

\[ \frac{dH}{de} \bigg|_{e=0} = -\int_{t_1}^{t_2} m\ddot{x}\eta \, dt - \int_{t_1}^{t_2} mg\eta \, dt = 0 \]

\[ -m\ddot{x} - mg = 0 \quad \rightarrow \quad -mg = m\ddot{x} \]
General solution for the functional

\[ H(x(t)) = \int_{t_1}^{t_2} (KE - PE) \, dt = \int_{t_1}^{t_2} L(x, \dot{x}, t) \, dt \]

\[ \frac{d}{dt} \left( \frac{dL}{d\dot{x}} \right) - \frac{dL}{dx} = 0 \]

The solution to the calculus of variation approach to minimize \( H(x) \)

Example: dynamics of a point mass

\[ L = \frac{1}{2} m\dot{x}^2 - mgx \]

\[ \frac{dL}{dx} = m\dot{x} \quad \frac{d}{dt} \left( \frac{dL}{dx} \right) = m\ddot{x} \quad \frac{dL}{dx} = -mg \]

\[ m\ddot{x} + mg = 0 \quad \rightarrow \quad -mg = m\ddot{x} \]

If there are external forces (from motors, muscles) acting on the system:

\[ \frac{d}{dt} \left( \frac{dL}{d\dot{x}} \right) - \frac{dL}{dx} = F \]

The primary problem in dynamics is to find an expression for the kinetic energy of the system.