Introduction to Inertial Dynamics

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Lecture notes on equations of motion of multi-joint robotic or biomechanical systems

What is dynamics?

A description of how forces acting on a system result in motion of that system.

Example: A ball of mass m is held 20 m off the ground. The force acting on the ball is the force of gravity: f=-mg where g=9.8 m/s^2. If we drop the ball, its dynamics are describe by:

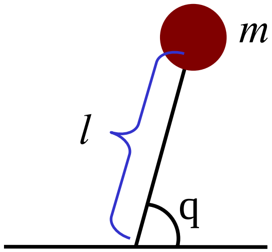
$$x(0) = 20\,\mathrm{m}$$

$$\dot{x}(0) = 0\,\mathrm{m/s}$$

$$f = -mg$$

$$\ddot{x}(t) = \frac{f}{m} = -g = -9.8 \quad \rightarrow \quad x(t) = 20 - 4.9t^2$$

Example: Dynamics of a single joint system with mass m, joint viscosity b, and length l.



$$\ddot{q} = \frac{1}{ml^2}(\tau - b\dot{q} - mgl\cos q)$$

Path of motion of a system is one that minimizes an energy cost

Imagine a point mass that is at position x_1 at time t_1 and ends up at position x_2 at t_2, for example: a ball falling from a height. The trajectory that it follows to get to x_2 is only one of an infinite number of pathways that it could have followed. But the point mass will always follow that same trajectory $x(t)$, given the same initial conditions. What is so special about the trajectory $x(t)$ that it actually does follow?

The trajectory $x(t)$ minimizes the following cost function:

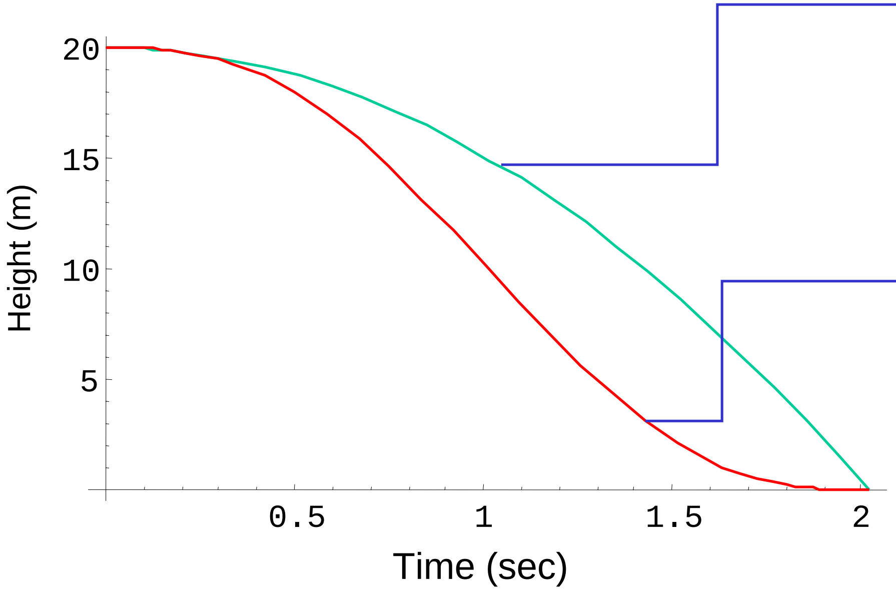
$$H(x(t)) = \int_{t_1}^{t_2} (KE - PE)\, dt = \int_{t_1}^{t_2} L(x, \dot{x}, t)\, dt$$

Kinetic energy

Potential energy

Lagrangian

For a point mass: $\begin{cases} KE = \frac{1}{2} m \dot{x}^2 \\ PE = mgx \end{cases}$



$$\ddot{x}(t) = -9.8 \quad \rightarrow \quad x(t) = 20 - 4.9t^2$$

$$H(x(t)) = \int_0^{2.02} \left(\frac{1}{2} m \dot{x}(t)^2 - mgx(t)\right) dt = -264$$

$$x(t) = 20 + 20(-1.21t^3 + 0.9t^4 - 0.178t^5)$$

$$H(x(t)) = \int_0^{2.02} \left(\frac{1}{2} m \dot{x}(t)^2 - mgx(t)\right) dt = -113$$

Solving the functional for a point mass

$$H(x(t)) = \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} (KE - PE)\,dt = \int_{t_1}^{t_2} \frac{1}{2}m\dot{x}^2 - mgx\,dt$$

$$x(t) \to x(t) + e\eta(t)$$

$$H(x + e\eta) = \int_{t_1}^{t_2} \frac{1}{2}m(\dot{x} + e\dot{\eta})^2\,dt - \int_{t_1}^{t_2} mg(x + e\eta)\,dt$$

$$H(x + e\eta) = \int_{t_1}^{t_2} \frac{1}{2}m(\dot{x} + e\dot{\eta})^2\,dt - \int_{t_1}^{t_2} mg(x + e\eta)\,dt$$

$$\left.\frac{dH}{de}\right|_{e=0} = \int_{t_1}^{t_2} m(\dot{x} + e\dot{\eta})\dot{\eta}\,dt - \int_{t_1}^{t_2} mg\eta\,dt$$

$$\left.\frac{dH}{de}\right|_{e=0} = \int_{t_1}^{t_2} m\dot{x}\dot{\eta}\,dt - \int_{t_1}^{t_2} mg\eta\,dt$$

$$u = \dot{x} \quad dv = \dot{\eta}\,dt \quad du = \ddot{x}dt \quad v = \eta$$

$$\int_{t_1}^{t_2} m\dot{x}\dot{\eta}\,dt = m\dot{x}\eta\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} m\eta\ddot{x}dt = -\int_{t_1}^{t_2} m\ddot{x}\eta dt$$

$$\left.\frac{dH}{de}\right|_{e=0} = -\int_{t_1}^{t_2} m\ddot{x}\eta dt - \int_{t_1}^{t_2} mg\eta\,dt = 0$$

$$-m\ddot{x} - mg = 0 \quad \to \quad \boxed{-mg = m\ddot{x}}$$

General solution for the functional

$$H(x(t)) = \int_{t_1}^{t_2} (KE - PE)\, dt = \int_{t_1}^{t_2} \underbrace{L(x, \dot{x}, t)}_{\text{Lagrangian}}\, dt$$

$$\boxed{\frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) - \frac{dL}{dx} = 0}$$

The solution to the calculus of variation approach to minimize $H(x)$

Example: dynamics of a point mass

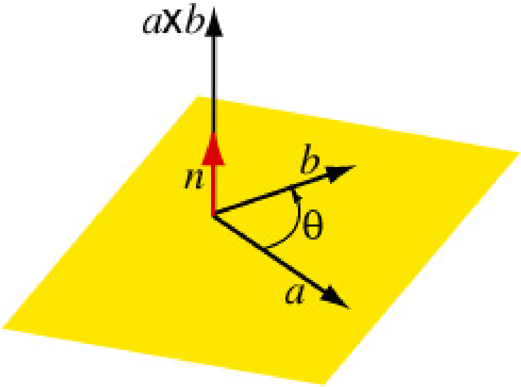
$$\left.\begin{aligned} &L = \frac{1}{2} m\dot{x}^2 - mgx \\ &\frac{dL}{d\dot{x}} = m\dot{x} \quad \frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) = m\ddot{x} \quad \frac{dL}{dx} = -mg \end{aligned}\right\} \quad m\ddot{x} + mg = 0 \;\; \rightarrow \;\; -mg = m\ddot{x}$$

If there are external forces (from motors, muscles) acting on the system:

$$\frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) - \frac{dL}{dx} = F$$

The primary problem in dynamics is to find an expression for the kinetic energy of the system.

Cross product of two vectors



Vectors a and b define a plane.

Vector n is perpendicular to that plane.

The angle between a and b is θ.

The magnitude of n increases like a screw rising out of the plane when we rotate vector a to reach vector b.

$$a \equiv \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad b \equiv \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Note that when a and b are parallel, a x b =0

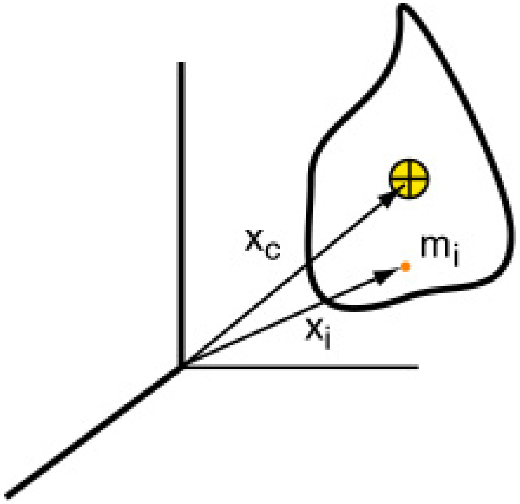
Center of mass of a rigid body

Assume that an object is composed of small "particles" of mass call m_i.
Total mass of the object is:

$$m \equiv \sum_i m_i$$

Position of particle i is specified by vector x_i

Center of mass x_c is the weighted average position of the particles.



$$x_c \equiv \frac{1}{m} \sum_i m_i x_i$$

Linear momentum of a particle in motion

Once an object starts moving, linear momentum describes the tendency for objects to continue moving in the same direction.

Momentum refers to mass times velocity of a particle.

Momentum of particle i :

$$p_i \equiv m_i \dot{x}_i$$

Here, because *m* is a scalar, velocity and momentum are vectors that are in the same direction.

Linear momentum of a rigid body in motion

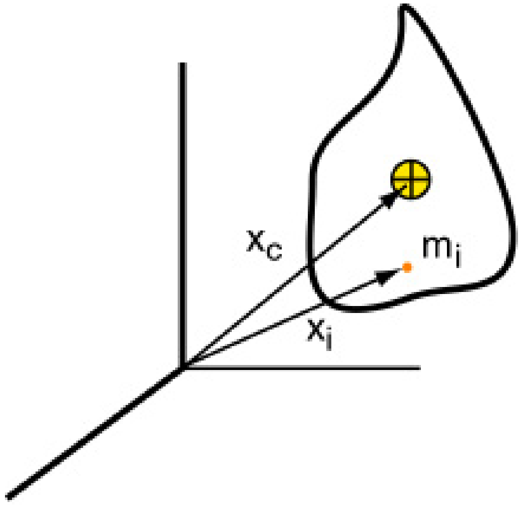
$$p \equiv \sum_i p_i$$

$$x_c \equiv \frac{1}{m} \sum_i m_i x_i$$

$$\frac{d}{dt}(x_c) = \frac{1}{m} \frac{d}{dt}\left(\sum_i m_i x_i \right)$$

$$\dot{x}_c = \frac{1}{m} \sum_i m_i \dot{x}_i$$

$$\dot{x}_c = \frac{1}{m} \sum_i p_i$$



$$\boxed{m\dot{x}_c = p}$$

Linear momentum of an object is equal to the total mass times the velocity at the center of mass of the object.

Force and change in linear momentum

If an object has momentum *p*, it will continue to have that momentum unless a force acts on it.

The rate of change in linear momentum is equal to the force.

The particle with mass m is being affected by force f and has velocity $\dot{x}$

$$p = m\dot{x}$$

$$\boxed{f \equiv \frac{dp}{dt} = m\ddot{x}}$$

Force is the rate of change of linear momentum.

The concepts of angular momentum and torque

Angular momentum is a measure of an object’s rotational motion. It describes the tendency for objects to continue spinning about their particular axis.

Inertia

$$h = Iw$$ Angular velocity

Angular momentum

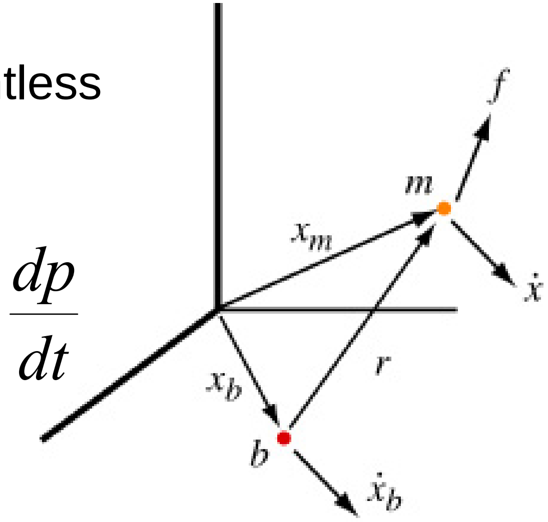
If an object has angular momentum *h*, it will continue to have that momentum unless a torque acts on it.

Torque is the rate of change of angular momentum.

Inertia

$$\tau \equiv \frac{dh}{dt} = I\alpha$$ Angular acceleration

Imagine that the particle is connected to point b with weightless rod of length r.



$$r \times f = r \times \frac{dp}{dt} \qquad \text{but note}: \frac{d}{dt}(r \times p) = \frac{dr}{dt} \times p + r \times \frac{dp}{dt}$$

$$= \frac{d}{dt}(r \times p) - \frac{dr}{dt} \times p$$

$$= \frac{d}{dt}(r \times p) - (\dot{x} - \dot{x}_b) \times p = \frac{d}{dt}(r \times p) - \underbrace{\dot{x} \times p}_{\text{0 because the two vectors are always parallel for a point mass}} + \dot{x}_b \times p$$

$$r \times f = \frac{d}{dt}(r \times p) + \dot{x}_b \times p$$

$$\text{If point } b \text{ is fixed}: \quad \dot{x}_b = 0 \rightarrow \quad \overbrace{r \times f}^{\text{torque}} = \frac{d}{dt}\overbrace{(r \times p)}^{\text{angular momentum of the particle}}$$

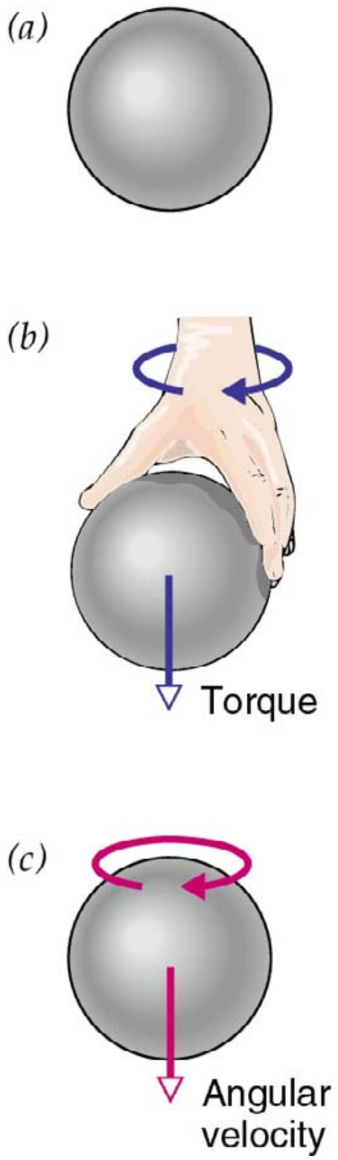
Note that if point b moves, torque on that point will depend on how we define velocity of point b.

Torque and angular velocity

(a) If you start with a ball that is not spinning,

(b) and you twist it with a torque

(c) the ball will have an angular velocity that is in the same direction as the torque vector.



Bloomfield, ***How Things Work*** **2001**

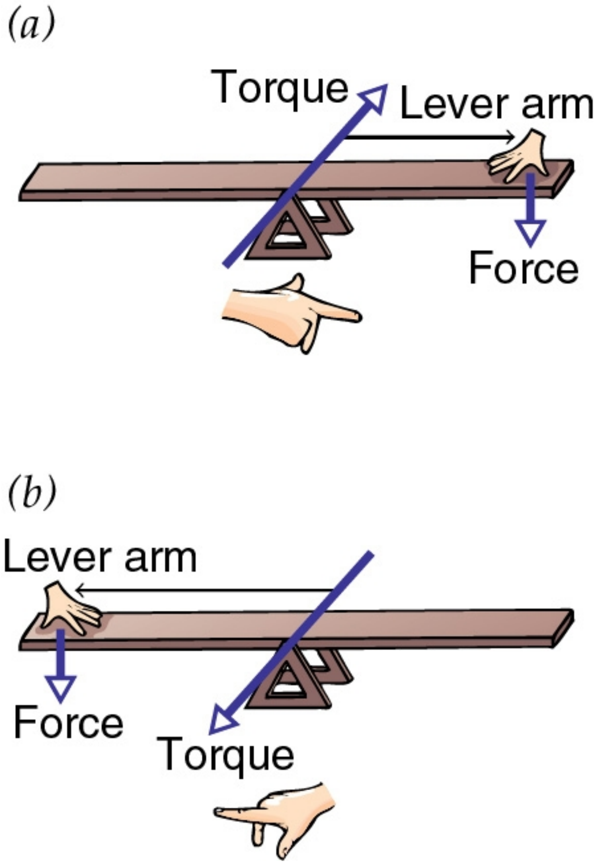
Torque and lever arms

$$\tau = r \times f$$

The torque on the seesaw obeys the “right hand rule”.

If the index finger points along the lever arm and the middle finger points along the force, the thumb points along the torque.

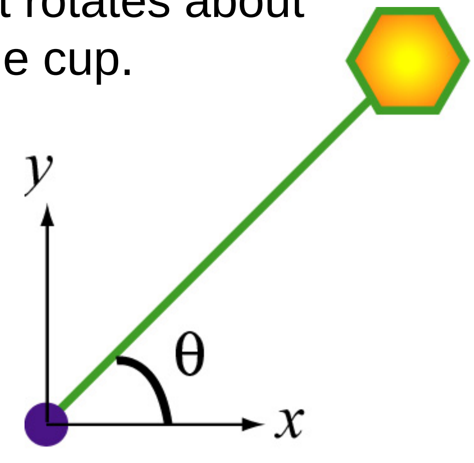
When you cut cardboard with a pair of scissors, it is best to move the cardboard as close as possible to the scissors’ pivot. Why?



Bloomfield, ***How Things Work*** **2001**

Centripetal forces

In an amusement park ride, a large “cup” can hold a child as it rotates about a center point. Objective: to describe the forces that act on the cup.



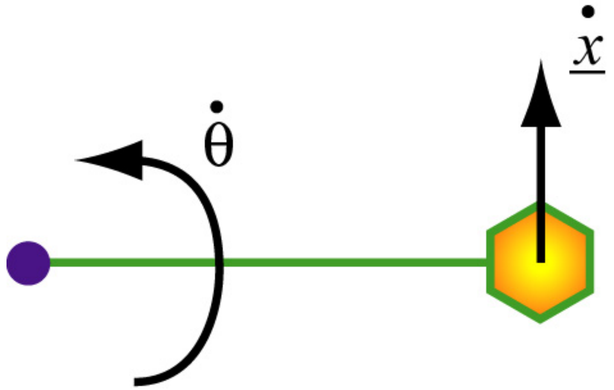
$$\left.\begin{aligned} KE &= \frac{1}{2}m\dot{\theta}^2 \\ PE &= 0 \end{aligned}\right\} L = KE + PE$$

$$\tau = \frac{d}{dt}\left(\frac{dL}{d\dot{\theta}}\right) + \frac{dL}{d\theta} = m\ddot{\theta}$$

$$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\underline{\dot{x}} = \frac{d\,\underline{x}}{dt} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} \frac{dx}{d\theta}\frac{d\theta}{dt} \\ \frac{dy}{d\theta}\frac{d\theta}{dt} \end{bmatrix} = \begin{bmatrix} -\dot{\theta}\sin\theta \\ \dot{\theta}\cos\theta \end{bmatrix}$$

linear velocity due to angular velocity



$$\underline{\dot{x}} = \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}-\dot{\theta}\sin\theta\\ \dot{\theta}\cos\theta\end{bmatrix} \qquad \underline{\ddot{x}} = \begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} = \begin{bmatrix}\dfrac{d\dot{x}}{dt}\\ \dfrac{d\dot{y}}{dt}\end{bmatrix}$$

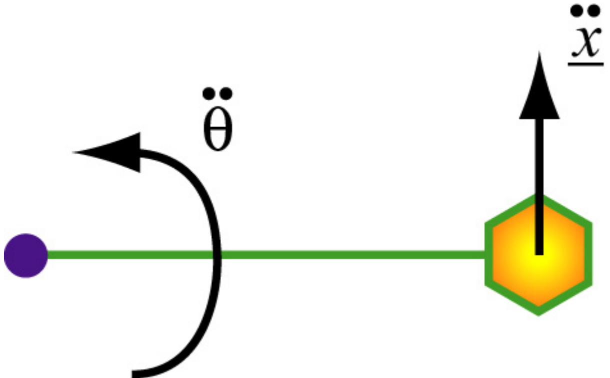
$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d}{dt}\left(-\dot{\theta}\sin\theta\right) = \frac{d}{dt}\left(-\dot{\theta}\right)\sin\theta - \dot{\theta}\frac{d}{dt}\left(\sin\theta\right)$$

$$= -\ddot{\theta}\sin\theta - \dot{\theta}\frac{d}{dt}\left(\sin\theta\right) = -\ddot{\theta}\sin\theta - \dot{\theta}\frac{d}{d\theta}\left(\sin\theta\right)\frac{d\theta}{dt}$$

$$= -\ddot{\theta}\sin\theta - \dot{\theta}\dot{\theta}\cos\theta = -\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta$$

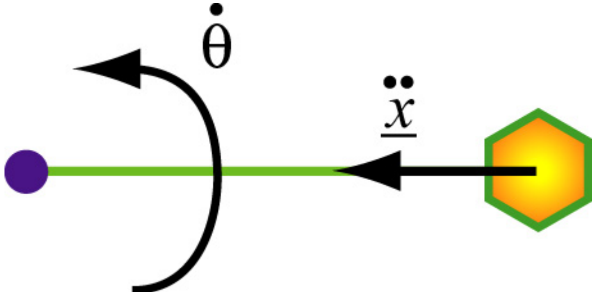
$$\ddot{y} = \frac{d\dot{y}}{dt} = \frac{d}{dt}\left(\dot{\theta}\cos\theta\right) = \ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta$$

$$\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}\ddot{\theta} + \begin{bmatrix}-\cos\theta\\ -\sin\theta\end{bmatrix}\dot{\theta}^2$$



linear acceleration due to angular acceleration

linear acceleration due to angular velocity



$$\left.\begin{aligned} KE &= \frac{1}{2} m \underline{\dot{x}}^T \, \underline{\dot{x}} \\ PE &= 0 \end{aligned}\right\} \; L = KE + PE = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right)$$

$$\underline{f} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \frac{d}{dt}\left(\frac{dL}{d\,\underline{\dot{x}}}\right) + \frac{dL}{d\,\underline{x}} = \frac{d}{dt}\begin{bmatrix} \frac{dL}{d\dot{x}} \\ \frac{dL}{d\dot{y}} \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} m\dot{x} \\ m\dot{y} \end{bmatrix} = m\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix}$$

$$\underline{f} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = m\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = m\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}\ddot{\theta} + m\begin{bmatrix} -\cos\theta \\ -\sin\theta \end{bmatrix}\dot{\theta}^2$$

What it means: If I would like the cup to rotate with velocity $\dot{\theta}(t)$ and acceleration $\ddot{\theta}(t)$, I need to produce force $\underline{f}(t)$.

$$\underline{f} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = m\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}\ddot{\theta} + \underbrace{m\begin{bmatrix} -\cos\theta \\ -\sin\theta \end{bmatrix}\dot{\theta}^2}_{\text{centripetal force}}$$

$$\underline{f} \underbrace{- m\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}\ddot{\theta} - m\begin{bmatrix} -\cos\theta \\ -\sin\theta \end{bmatrix}\dot{\theta}^2}_{\text{force that the mass of the cup “produces” to resist motion}} = 0$$

force that my torque
motor must make to
produce motion

The child in the cup feels the force “produced” by his mass: she feels a centripetal force pushing her outward as she rotates.

Gravity also produces a force on our body. Simulators use gravity to fool the brain into thinking it is feeling a force due to motion.

Torque about center of mass of a rigid body

$$ {}^{c}\tau = \sum_i r_i \times f_i = \sum_i r_i \times \frac{dp_i}{dt} \quad \text{but note that :} \ \frac{d}{dt}(r_i \times p_i) = \frac{dr_i}{dt} \times p_i + r_i \times \frac{dp_i}{dt} $$

$$ = \sum_i \frac{d}{dt}(r_i \times p_i) - \frac{dr_i}{dt} \times p_i $$

$$ = \sum_i \frac{d}{dt}(r_i \times p_i) - (\dot{x}_i - \dot{x}_c) \times p_i = \frac{d}{dt}\sum_i (r_i \times p_i) + \sum_i \dot{x}_c \times p_i $$

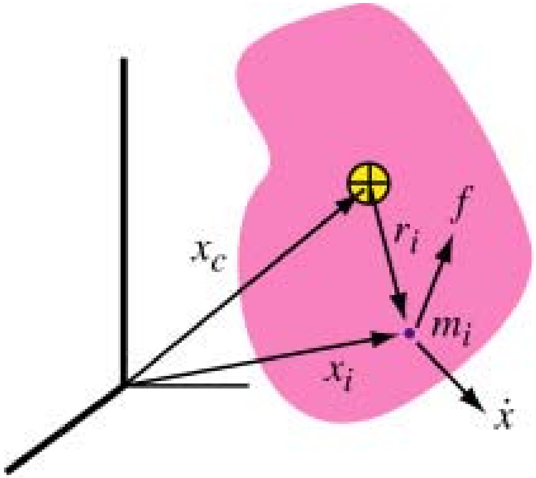
$$ = \frac{d}{dt}\sum_i (r_i \times p_i) + \dot{x}_c \times \sum_i p_i \quad \text{but note that :} \ p = \sum_i p_i = m\dot{x}_c $$

$$ {}^{c}\tau = \frac{d}{dt}\sum_i (r_i \times p_i) + \dot{x}_c \times m\dot{x}_c $$

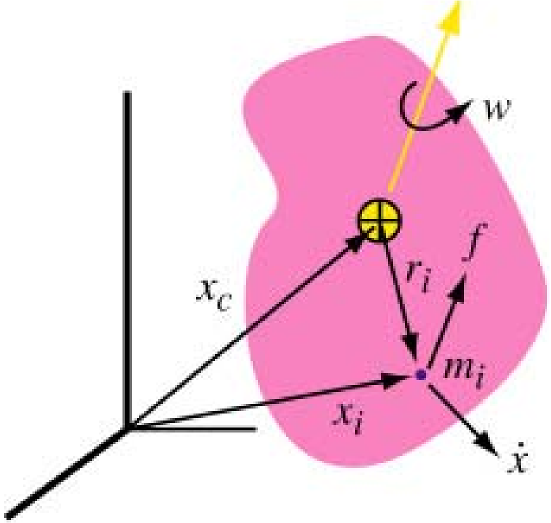
$$ \underbrace{{}^{c}\tau} = \frac{d}{dt}\underbrace{\sum_i (r_i \times p_i)} $$

Torque about center of mass

Angular momentum of a rigid body at its center of mass



Angular momentum about center of mass of a rigid body



$$^c\tau = \frac{d}{dt}\sum_i (r_i \times p_i)$$

$$^c h \equiv \sum_i r_i \times p_i = \sum_i r_i \times m_i \dot{x}_i \qquad \text{we have}: \ \dot{x}_i = \dot{x}_c + \underset{\downarrow}{w} \times r_i$$

Angular velocity of the body
about its center of mass

$$^c h = \sum_i r_i \times m_i (\dot{x}_c + w \times r_i)$$

$$^c h = \sum_i m_i r_i \times \dot{x}_c + \sum_i m_i r_i \times (w \times r_i)$$

$$\text{we have}: \ \sum_i m_i r_i = \sum_i m_i (x_i - x_c) \ \text{ and } x_c = \frac{1}{m}\sum_j m_j x_j$$

$$\sum_i m_i r_i = \sum_i m_i (x_i - \frac{1}{m}\sum_j m_j x_j) = \sum_i m_i x_i - \frac{\sum m_i}{m}\sum_j m_j x_j = 0$$

$$\boxed{^c h = \sum_i m_i r_i \times (w \times r_i)}$$

Recall that: Torque is the rate of change of angular momentum.

$$\tau \equiv \frac{dh}{dt} = \overset{\substack{\text{Inertia}\\ \downarrow}}{I}\alpha \longleftarrow \text{Angular acceleration}$$

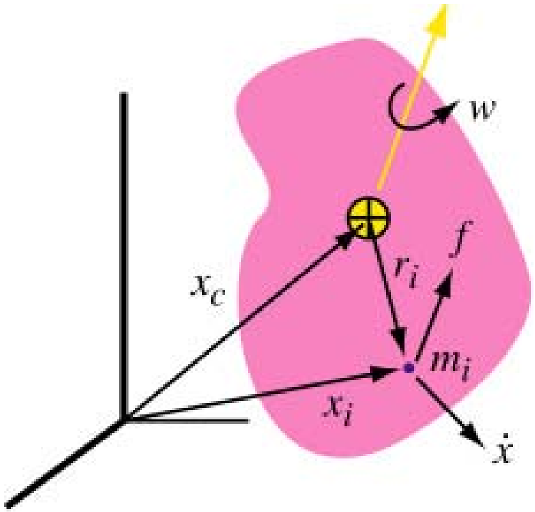
$$h = Iw \longleftarrow \text{Angular velocity}$$

To define inertia, we need to be able to write angular momentum (a vector) in terms of something (inertia, which is a matrix) that multiplies angular velocity.

$${}^{c}h = \sum_{i} m_i r_i \times (w \times r_i)$$

$$h = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = Iw = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

Inertia about center of mass of a rigid body



$${}^{c}h = \sum_i m_i r_i \times (w \times r_i)$$

use identity $a \times (b \times c) = (a \cdot c)b - (b \cdot a)c$

$${}^{c}h = \sum_i m_i\big((r_i \cdot r_i)w - (w \cdot r_i)r_i\big) = \sum_i m_i\left(|r_i|^2 w - (w \cdot r_i)r_i\right)$$

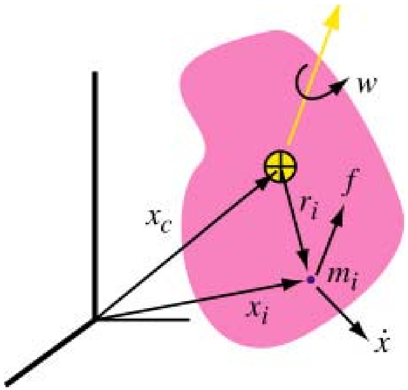
$${}^{c}h = \begin{bmatrix} {}^{c}h_x \\ {}^{c}h_y \\ {}^{c}h_z \end{bmatrix} \quad w = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} \quad r_i = \begin{bmatrix} r_{ix} \\ r_{iy} \\ r_{iz} \end{bmatrix}$$

$$w \cdot r_i = w_x r_{ix} + w_y r_{iy} + w_z r_{iz}$$

$$(w \cdot r_i)r_i = \begin{bmatrix} (w_x r_{ix} + w_y r_{iy} + w_z r_{iz})r_{ix} \\ (w_x r_{ix} + w_y r_{iy} + w_z r_{iz})r_{iy} \\ (w_x r_{ix} + w_y r_{iy} + w_z r_{iz})r_{iz} \end{bmatrix}$$

$${}^{c}h = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \begin{bmatrix} \sum_i m_i(|r_i|^2 w_x - r_{ix}^2 w_x - w_y r_{ix} r_{iy} - w_z r_{ix} r_{iz}) \\ \sum_i m_i(-w_x r_{ix} r_{iy} + |r_i|^2 w_y - r_{iy}^2 w_y - w_z r_{iy} r_{iz}) \\ \sum_i m_i(-w_x r_{ix} r_{iz} - w_y r_{iy} r_{iz} + |r_i|^2 w_z - r_{iz}^2 w_z) \end{bmatrix}$$

Inertia about center of mass of a rigid body



$$ {}^c h = \begin{bmatrix} {}^c h_x \\ {}^c h_y \\ {}^c h_z \end{bmatrix} = \begin{bmatrix} \sum_i m_i (|r_i|^2 w_x - r_{ix}^2 w_x - w_y r_{ix} r_{iy} - w_z r_{ix} r_{iz}) \\ \sum_i m_i (-w_x r_{ix} r_{iy} + |r_i|^2 w_y - r_{iy}^2 w_y - w_z r_{iy} r_{iz}) \\ \sum_i m_i (-w_x r_{ix} r_{iz} - w_y r_{iy} r_{iz} + |r_i|^2 w_z - r_{iz}^2 w_z) \end{bmatrix} $$

$$ |r_i|^2 = r_{ix}^2 + r_{iy}^2 + r_{iz}^2 \;\rightarrow\; |r_i|^2 - r_{ix}^2 = r_{iy}^2 + r_{iz}^2 $$

$$ {}^c h = \begin{bmatrix} \sum_i m_i (r_{iy}^2 + r_{iz}^2) w_x - \sum_i m_i r_{ix} r_{iy} w_y - \sum_i m_i r_{ix} r_{iz} w_z \\ -\sum_i m_i r_{ix} r_{iy} w_x + \sum_i m_i (r_{ix}^2 + r_{iz}^2) w_y - \sum_i m_i r_{iy} r_{iz} w_z \\ -\sum_i m_i r_{ix} r_{iz} w_x - \sum_i m_i r_{iy} r_{iz} w_y + \sum_i m_i (r_{ix}^2 + r_{iy}^2) w_z \end{bmatrix} $$

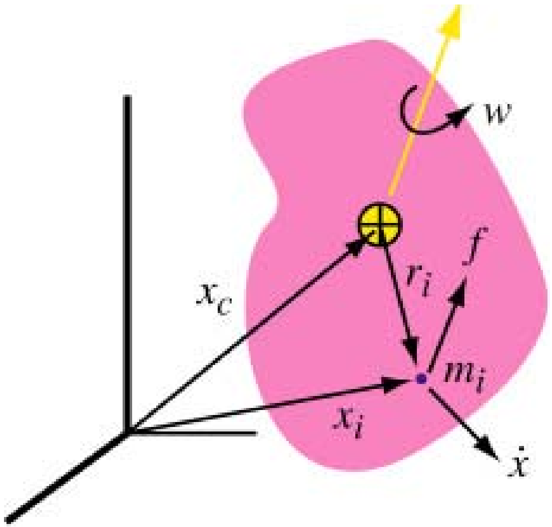
Angular velocity about center of mass

Inertia about center of mass

$$ {}^c h = {}^c I w = \begin{bmatrix} \sum_i m_i (r_{iy}^2 + r_{iz}^2) & -\sum_i m_i r_{ix} r_{iy} & -\sum_i m_i r_{ix} r_{iz} \\ -\sum_i m_i r_{ix} r_{iy} & \sum_i m_i (r_{ix}^2 + r_{iz}^2) & -\sum_i m_i r_{iy} r_{iz} w_z \\ -\sum_i m_i r_{ix} r_{iz} & -\sum_i m_i r_{iy} r_{iz} & \sum_i m_i (r_{ix}^2 + r_{iy}^2) \end{bmatrix} \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} = \begin{bmatrix} {}^c I_{xx} & {}^c I_{xy} & {}^c I_{xz} \\ {}^c I_{yx} & {}^c I_{yy} & {}^c I_{yz} \\ {}^c I_{zx} & {}^c I_{zy} & {}^c I_{zz} \end{bmatrix} \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} $$

Inertia about center of mass of a rigid body

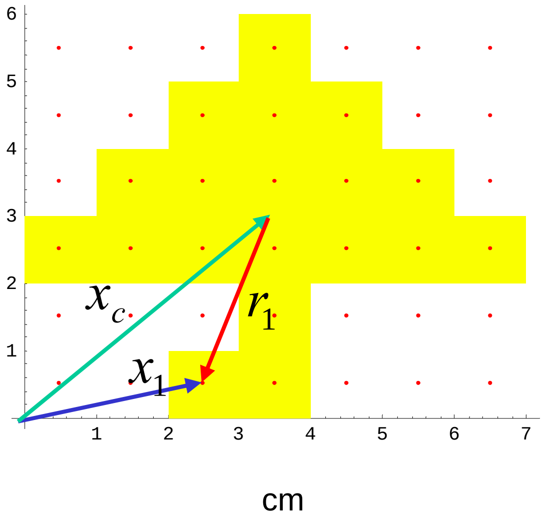
$$
{}^{c}I = \begin{bmatrix} \sum\limits_i m_i (r_{iy}^2 + r_{iz}^2) & -\sum\limits_i m_i r_{ix} r_{iy} & -\sum\limits_i m_i r_{ix} r_{iz} \\ -\sum\limits_i m_i r_{ix} r_{iy} & \sum\limits_i m_i (r_{ix}^2 + r_{iz}^2) & -\sum\limits_i m_i r_{iy} r_{iz} w_z \\ -\sum\limits_i m_i r_{ix} r_{iz} & -\sum\limits_i m_i r_{iy} r_{iz} & \sum\limits_i m_i (r_{ix}^2 + r_{iy}^2) \end{bmatrix}
$$



Note that the inertia matrix, when written about the center of mass, is always symmetric.

Example:

The object shown here is flat and thin. It is made of aluminum and weighs 19 grams, and is uniform. Our task is to find its center of mass and its inertia about its center of mass.



- Divide up the piece into equal size squares of 1x1 cm.
- Each small square is assumed to be a particle of mass

1 with its location specified by the center of square.

$$x_1 = \begin{bmatrix} 2.5 \\ 0.5 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 3.5 \\ 0.5 \\ 0 \end{bmatrix} \ldots$$

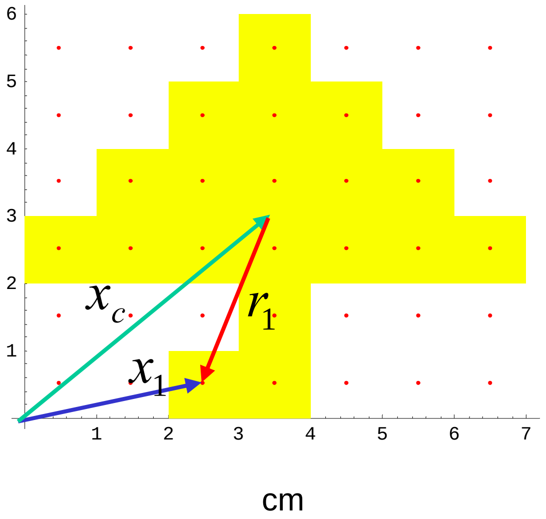
$$m_1 = 1 \qquad m_2 = 1 \qquad \ldots$$

$$x_c = \frac{1}{m}\sum_i m_i x_i = \frac{1}{19}\begin{bmatrix} 65.5 \\ 56.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.45 \\ 2.97 \\ 0 \end{bmatrix} \text{cm}$$

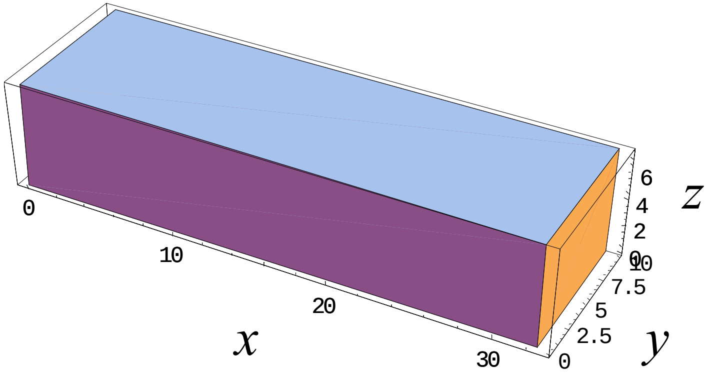
$$r_1 = x_1 - x_c = \begin{bmatrix} -0.95 \\ -2.47 \\ 0 \end{bmatrix} \quad r_2 = \begin{bmatrix} 0.05 \\ -2.47 \\ 0 \end{bmatrix} \ldots$$

$${}^{c}I_{xx} = \sum_i m_i (r_{iy}^2 + r_{iz}^2) = 30.7$$

$${}^{c}I = \begin{bmatrix} {}^{c}I_{xx} & {}^{c}I_{xy} & {}^{c}I_{xz} \\ {}^{c}I_{yx} & {}^{c}I_{yy} & {}^{c}I_{yz} \\ {}^{c}I_{zx} & {}^{c}I_{zy} & {}^{c}I_{zz} \end{bmatrix} = \begin{bmatrix} 30.7 & -2.5 & 0 \\ -2.5 & 40.9 & 0 \\ 0 & 0 & 71.7 \end{bmatrix} \text{g cm}^2$$



Inertia of a cuboid model of the human upper arm



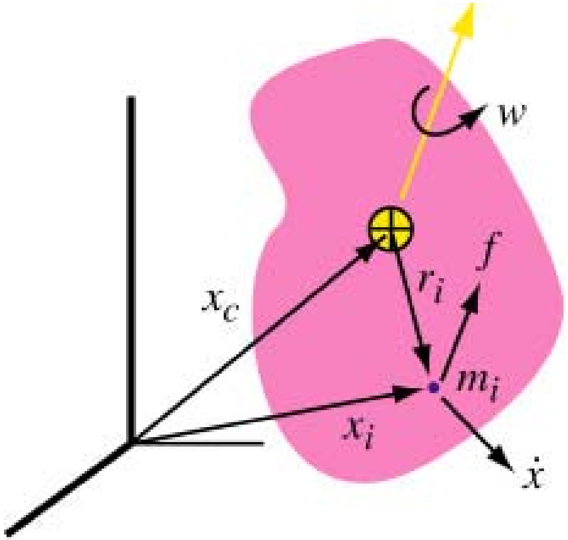
Size of the upper arm: 32.5x10x7.5 cm
density: 1.05 g/cm^3
“particle” size: 2.5x2.5x2.5 cm

$$m = 2560$$

$$x_c = \begin{bmatrix} 16.25 \\ 5.0 \\ 3.75 \end{bmatrix}$$

$$^c I = \begin{bmatrix} 30.7 & 0 & 0 \\ 0 & 234.6 & 0 \\ 0 & 0 & 243.9 \end{bmatrix} \text{kg cm}^2$$

Kinetic energy of a rigid body in motion



$$KE = \frac{1}{2}\sum_i m_i \dot{x}_i \cdot \dot{x}_i$$

$$\dot{x}_i = \dot{x}_c + w \times r_i$$

$$KE = \frac{1}{2}\sum_i m_i \left(\dot{x}_c \cdot \dot{x}_c + 2\dot{x}_c \cdot w \times r_i + w \times r_i \cdot w \times r_i\right)$$

$$\text{we had}: \sum_i m_i r_i = 0$$

$$KE = \frac{1}{2}\sum_i m_i \dot{x}_c \cdot \dot{x}_c + \frac{1}{2}\sum_i m_i w \times r_i \cdot w \times r_i$$

$$\text{we have}: \quad (a \times b) \cdot c = a \cdot (b \times c)$$

$$w \times r_i \cdot w \times r_i = w \cdot (r_i \times w \times r_i)$$

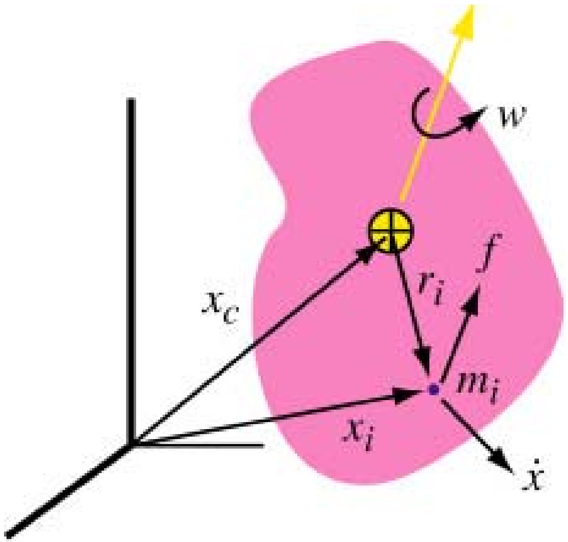
$$KE = \frac{1}{2}\sum_i m_i \dot{x}_c \cdot \dot{x}_c + \frac{1}{2}\sum_i m_i w \cdot (r_i \times w \times r_i)$$

$$\text{we had}: \ {}^c h = \sum_i m_i r_i \times (w \times r_i) \quad \text{and} \quad {}^c h = {}^c I w$$

$$KE = \frac{1}{2}\sum_i m_i \dot{x}_c \cdot \dot{x}_c + \frac{1}{2} w \cdot {}^c h$$

$$\boxed{KE = \frac{1}{2} m \dot{x}_c^T \dot{x}_c + \frac{1}{2} w^T \ {}^c I w}$$

Kinetic energy of a rigid body in motion



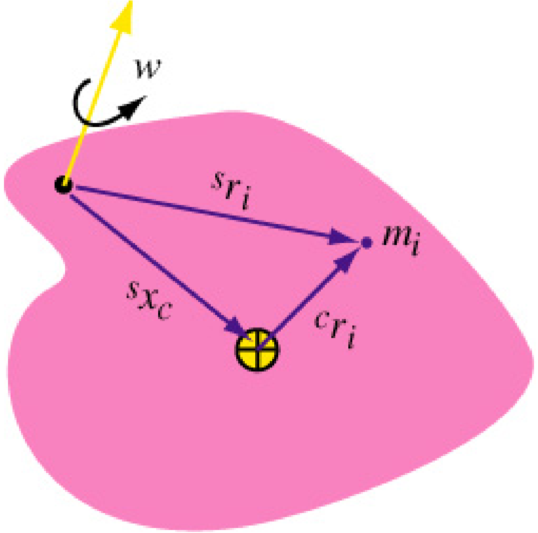
$$KE = \frac{1}{2} m \dot{x}_c^T \dot{x}_c + \frac{1}{2} w^T \ {}^{c}I w$$

KE due to translation at the center of mass

KE due to rotation at the center of mass

Inertia about a point other than center of mass

When the object is rotating at its center of mass:



$$^c h = {}^c I w = \begin{bmatrix} ^c I_{xx} & ^c I_{xy} & ^c I_{xz} \\ ^c I_{yx} & ^c I_{yy} & ^c I_{yz} \\ ^c I_{zx} & ^c I_{zy} & ^c I_{zz} \end{bmatrix} \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

for example : $^c I_{xx} = \sum_i m_i (^c r_{iy}^2 + {}^c r_{iz}^2)$

When rotation is occurring about another point on the object:

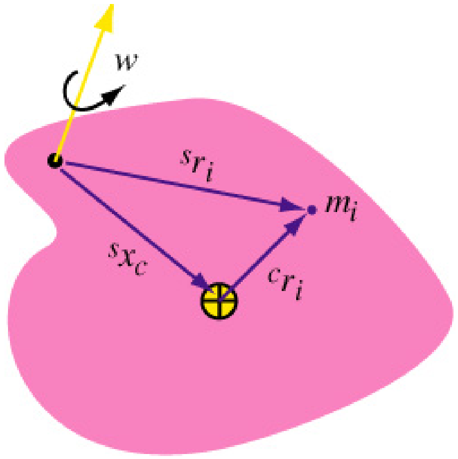
$$^s h = {}^s I w = \begin{bmatrix} ^s I_{xx} & ^s I_{xy} & ^s I_{xz} \\ ^s I_{yx} & ^s I_{yy} & ^s I_{yz} \\ ^s I_{zx} & ^s I_{zy} & ^s I_{zz} \end{bmatrix} \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

$$^s I_{xx} = \sum_i m_i (^s r_{iy}^2 + {}^s r_{iz}^2)$$

$$^s r_i \equiv \begin{bmatrix} ^s r_{ix} \\ ^s r_{iy} \\ ^s r_{iz} \end{bmatrix} = {}^s x_c + {}^c r_i = \begin{bmatrix} ^s x_{cx} \\ ^s x_{cy} \\ ^s x_{cz} \end{bmatrix} + \begin{bmatrix} ^c r_{ix} \\ ^c r_{iy} \\ ^c r_{iz} \end{bmatrix}$$

$$^s r_{iy}^2 = \left(^s x_{cy} + {}^c r_{iy} \right)^2 \quad {}^s r_{iz}^2 = \left(^s x_{cz} + {}^c r_{iz} \right)^2$$

$$^s I_{xx} = \sum_i m_i (^c r_{iy}^2 + {}^c r_{iz}^2) + \sum_i m_i (^s x_{cy}^2 + {}^s x_{cz}^2 + 2 {}^s x_{cy} {}^c r_{iy} + 2 {}^s x_{cz} {}^c r_{iz})$$



$${}^sI_{xx}=\sum_i m_i({}^cr_{iy}^2+{}^cr_{iz}^2)+\sum_i m_i({}^sx_{cy}^2+{}^sx_{cz}^2+2\,{}^sx_{cy}\,{}^cr_{iy}+2\,{}^sx_{cz}\,{}^cr_{iz})$$

we had : $\sum_i m_i\,{}^cr_{iy}=\sum_i m_i\,{}^cr_{iz}=0$

$${}^sI_{xx}={}^cI_{xx}+\sum_i m_i({}^sx_{cy}^2+{}^sx_{cz}^2)$$

$${}^sI_{xy}=-\sum_i m_i\,{}^sr_{ix}\,{}^sr_{iy}=-\sum_i m_i\,{}^cr_{ix}\,{}^cr_{iy}-\sum_i m_i\,{}^sx_{cx}\,{}^sx_{cy}=I_{cxy}-\sum_i m_i\,{}^sx_{cx}\,{}^sx_{cy}$$

$${}^sI_{xz}=-\sum_i m_i\,{}^sr_{ix}\,{}^sr_{iz}=I_{cxz}-\sum_i m_i\,{}^sx_{cx}\,{}^sx_{cz}$$

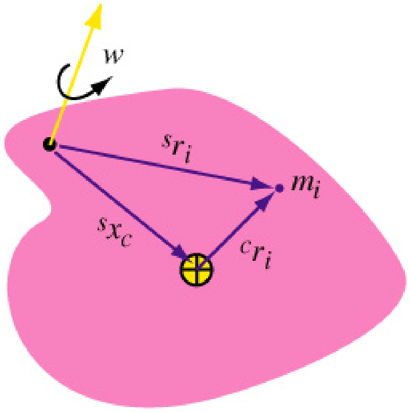
$${}^sI_{yy}=\sum_i m_i({}^sr_{ix}^2+{}^sr_{iz}^2)=I_{cyy}+\sum_i m_i({}^sx_{cx}^2+{}^sx_{cz}^2)$$

$${}^sI_{yz}=-\sum_i m_i\,{}^sr_{iy}\,{}^sr_{iz}=I_{cyz}-\sum_i m_i\,{}^sx_{cy}\,{}^sx_{cz}$$

$${}^sI_{zz}=\sum_i m_i({}^sr_{ix}^2+{}^sr_{iy}^2)=I_{czz}+\sum_i m_i({}^sx_{cx}^2+{}^sx_{cy}^2)$$

$$\boxed{{}^sI={}^cI+m\begin{bmatrix}{}^sx_{cy}^2+{}^sx_{cz}^2 & -{}^sx_{cx}\,{}^sx_{cy} & -{}^sx_{cx}\,{}^sx_{cz}\\ -{}^sx_{cx}\,{}^sx_{cy} & {}^sx_{cx}^2+{}^sx_{cz}^2 & -{}^sx_{cy}\,{}^sx_{cz}\\ -{}^sx_{cx}\,{}^sx_{cz} & -{}^sx_{cy}\,{}^sx_{cz} & {}^sx_{cx}^2+{}^sx_{cy}^2\end{bmatrix}}$$

Properties of the inertia matrix



$$
{}^{c}I = \begin{bmatrix} \sum_i m_i({}^{c}r_{iy}^2 + {}^{c}r_{iz}^2) & -\sum_i m_i\,{}^{c}r_{ix}\,{}^{c}r_{iy} & -\sum_i m_i\,{}^{c}r_{ix}\,{}^{c}r_{iz} \\ -\sum_i m_i\,{}^{c}r_{ix}\,{}^{c}r_{iy} & \sum_i m_i(r_{ix}^2 + r_{iz}^2) & -\sum_i m_i\,{}^{c}r_{iy}\,{}^{c}r_{iz} \\ -\sum_i m_i\,{}^{c}r_{ix}\,{}^{c}r_{iz} & -\sum_i m_i\,{}^{c}r_{iy}\,{}^{c}r_{iz} & \sum_i m_i({}^{c}r_{ix}^2 + {}^{c}r_{iy}^2) \end{bmatrix}
$$

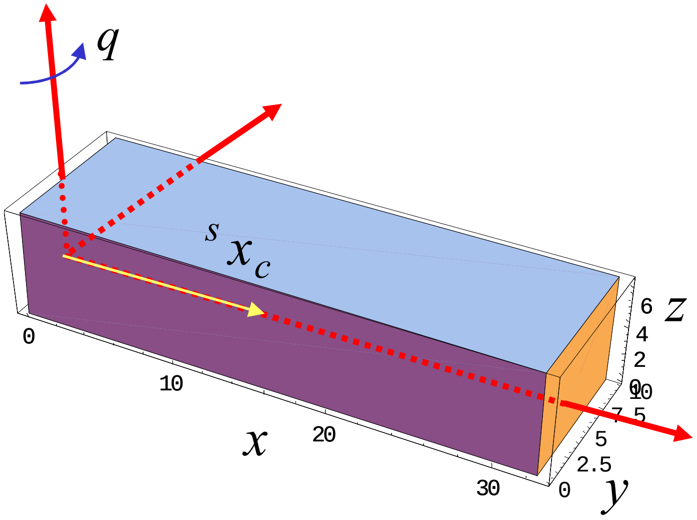
$$
{}^{s}I = {}^{c}I + m \begin{bmatrix} {}^{s}x_{cy}^2 + {}^{s}x_{cz}^2 & -{}^{s}x_{cx}\,{}^{s}x_{cy} & -{}^{s}x_{cx}\,{}^{s}x_{cz} \\ -{}^{s}x_{cx}\,{}^{s}x_{cy} & {}^{s}x_{cx}^2 + {}^{s}x_{cz}^2 & -{}^{s}x_{cy}\,{}^{s}x_{cz} \\ -{}^{s}x_{cx}\,{}^{s}x_{cz} & -{}^{s}x_{cy}\,{}^{s}x_{cz} & {}^{s}x_{cx}^2 + {}^{s}x_{cy}^2 \end{bmatrix}
$$

Inertia remains symmetric even if we express it about a point other than center of mass.

Mass appears linearly in all expressions of inertia.

When the body is rotating, the vector ${}^{s}x_c$ will have to be written in terms of angular position of the body with respect to the point of rotation. Therefore, I_s will be a function of position of the body.

Cuboid model of the upper arm rotating about the shoulder



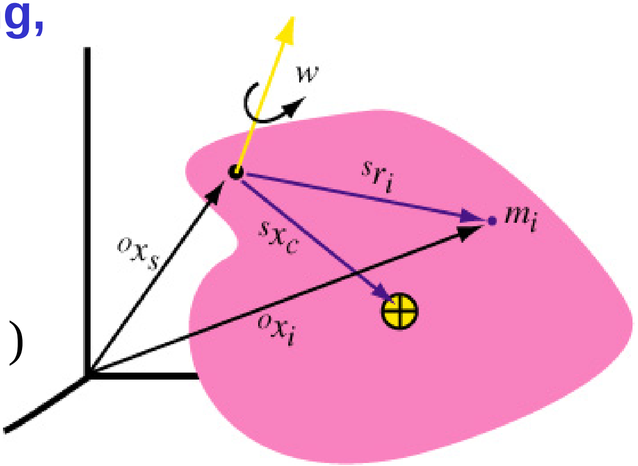
$$
{}^sI = {}^cI + m\begin{bmatrix} {}^sx_{cy}^2+{}^sx_{cz}^2 & -{}^sx_{cx}\,{}^sx_{cy} & -{}^sx_{cx}\,{}^sx_{cz} \\ -{}^sx_{cx}\,{}^sx_{cy} & {}^sx_{cx}^2+{}^sx_{cz}^2 & -{}^sx_{cy}\,{}^sx_{cz} \\ -{}^sx_{cx}\,{}^sx_{cz} & -{}^sx_{cy}\,{}^sx_{cz} & {}^sx_{cx}^2+{}^sx_{cy}^2 \end{bmatrix}
$$

$$
m = 2560 \text{ g}
$$

$$
{}^sx_c \equiv \begin{bmatrix} {}^sx_{cx} \\ {}^sx_{cy} \\ {}^sx_{cz} \end{bmatrix} = \begin{bmatrix} 16.25\cos q \\ 16.25\sin q \\ 0 \end{bmatrix} cm
$$

$$
{}^sI = \begin{bmatrix} 30.7 & 0 & 0 \\ 0 & 234.6 & 0 \\ 0 & 0 & 243.9 \end{bmatrix} + 2.560\begin{bmatrix} 264\sin^2 q & -264\cos q\sin q & 0 \\ -264\cos q\sin q & 264\cos^2 q & 0 \\ 0 & 0 & 264 \end{bmatrix} \text{kg cm}^2
$$

Kinetic energy when body is translating and rotating, but rotation is not about its center of mass



$$KE = \frac{1}{2}\sum_i m_i\, {}^o\dot{x}_i \cdot {}^o\dot{x}_i \qquad {}^o\dot{x}_i = {}^o\dot{x}_s + w\times {}^s r_i$$

$$KE = \sum_i \frac{1}{2} m_i\, {}^o\dot{x}_s \cdot {}^o\dot{x}_s + m_i\, {}^o\dot{x}_s \cdot w\times {}^s r_i + \frac{1}{2} m_i (w\times {}^s r_i)\cdot(w\times {}^s r_i)$$

$$\frac{1}{2}\sum_i m_i\, {}^o\dot{x}_s \cdot {}^o\dot{x}_s = \frac{1}{2} m\, {}^o\dot{x}_s^T\, {}^o\dot{x}_s$$

we had: ${}^s x_c = \frac{1}{m}\sum_i m_i\, {}^s r_i \quad \rightarrow \quad \sum_i m_i\, {}^o\dot{x}_s \cdot w\times {}^s r_i = \sum_i {}^o\dot{x}_s \cdot w\times m_i\, {}^s r_i = m\, {}^o\dot{x}_s \cdot w\times {}^s x_c$

we had: $(a\times b)\cdot c = a\cdot(b\times c)$

$$\frac{1}{2}\sum_i m_i (w\times {}^s r_i)\cdot(w\times {}^s r_i) = \frac{1}{2}\sum_i w\cdot(m_i\, {}^s r_i \times (w\times {}^s r_i)) = \frac{1}{2} w^T\, {}^s I w$$

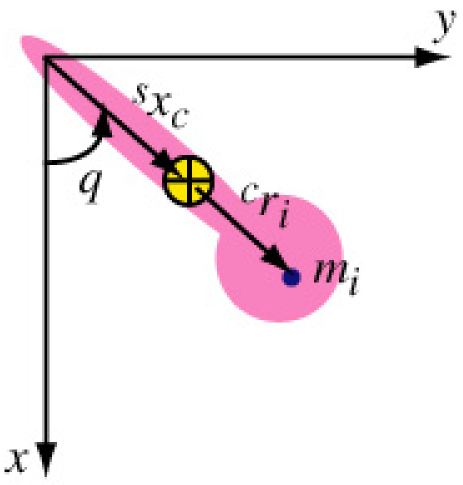
$$\boxed{KE = \frac{1}{2} m\, {}^o\dot{x}_s^T\, {}^o\dot{x}_s + \frac{1}{2} w^T\, {}^s I w + m\, {}^o\dot{x}_s^T (w\times {}^s x_c)}$$

KE due to translation

KE due to rotation

KE due to rotation caused by translation

Example: KE and dynamics of a pendulum

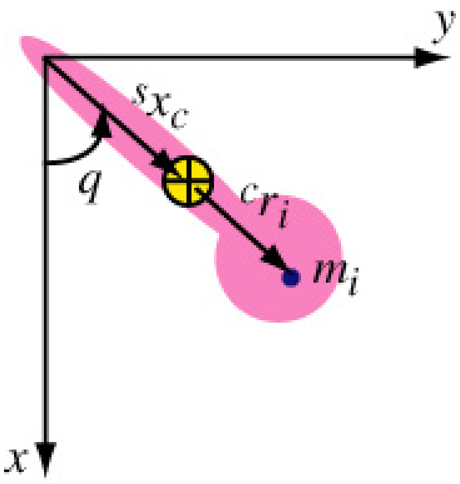


$$^cI = \begin{bmatrix} \sum_i m_i({}^c r_{iy}^2 + {}^c r_{iz}^2) & -\sum_i m_i {}^c r_{ix} {}^c r_{iy} & -\sum_i m_i {}^c r_{ix} {}^c r_{iz} \\ -\sum_i m_i {}^c r_{ix} {}^c r_{iy} & \sum_i m_i (r_{ix}^2 + r_{iz}^2) & -\sum_i m_i {}^c r_{iy} {}^c r_{iz} \\ -\sum_i m_i {}^c r_{ix} {}^c r_{iz} & -\sum_i m_i {}^c r_{iy} {}^c r_{iz} & \sum_i m_i({}^c r_{ix}^2 + {}^c r_{iy}^2) \end{bmatrix} = \begin{bmatrix} {}^cI_{xx} & {}^cI_{xy} & 0 \\ {}^cI_{yx} & {}^cI_{yy} & 0 \\ 0 & 0 & {}^cI_{zz} \end{bmatrix}$$

$$\dot{x}_s = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 0 \\ \dot{q} \end{bmatrix} \quad {}^sI = {}^cI + m \begin{bmatrix} {}^s x_{cy}^2 + {}^s x_{cz}^2 & -{}^s x_{cx} {}^s x_{cy} & -{}^s x_{cx} {}^s x_{cz} \\ -{}^s x_{cx} {}^s x_{cy} & {}^s x_{cx}^2 + {}^s x_{cz}^2 & -{}^s x_{cy} {}^s x_{cz} \\ -{}^s x_{cx} {}^s x_{cz} & -{}^s x_{cy} {}^s x_{cz} & {}^s x_{cx}^2 + {}^s x_{cy}^2 \end{bmatrix}$$

$$^s x_c = \begin{bmatrix} {}^s x_{cx} \\ {}^s x_{cy} \\ {}^s x_{cz} \end{bmatrix} = \begin{bmatrix} l_c \cos q \\ l_c \sin q \\ 0 \end{bmatrix} \text{ where } l_c = \left| {}^s x_c \right| \text{ (length of that vector)}$$

$$^sI = {}^cI + m \begin{bmatrix} \left|{}^s x_c\right|^2 \sin^2(q) & -\left|{}^s x_c\right|^2 \cos(q)\sin(q) & 0 \\ -\left|{}^s x_c\right|^2 \cos(q)\sin(q) & \left|{}^s x_c\right|^2 \cos^2(q) & 0 \\ 0 & 0 & \left|{}^s x_c\right|^2 \end{bmatrix}$$



$$ {}^sI = {}^cI + m\begin{bmatrix} \left|{}^sx_c\right|^2\sin^2(q) & -\left|{}^sx_c\right|^2\cos(q)\sin(q) & 0 \\ -\left|{}^sx_c\right|^2\cos(q)\sin(q) & \left|{}^sx_c\right|^2\cos^2(q) & 0 \\ 0 & 0 & \left|{}^sx_c\right|^2 \end{bmatrix} $$

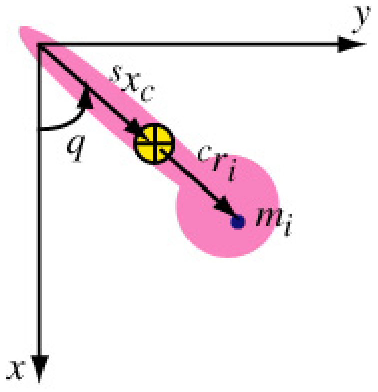
$$ KE = \frac{1}{2}m\dot{x}_s^T\dot{x}_s + \frac{1}{2}w^T\ {}^sIw + m\dot{x}_s^T(w\times\ {}^sx_c) $$

$$ KE = \frac{1}{2}w^T\ {}^sIw = \frac{1}{2}w^T\begin{bmatrix} 0 \\ 0 \\ (m\left|{}^sx_c\right|^2 + {}^cI_{zz})\dot{q} \end{bmatrix} = \frac{1}{2}(m\left|{}^sx_c\right|^2 + {}^cI_{zz})\dot{q}^2 $$

$$ w = \begin{bmatrix} 0 \\ 0 \\ \dot{q} \end{bmatrix} $$

$$ PE = mass.gravity.height = mg\left|{}^sx_c\right|(1-\cos(q)) $$

$$ L = KE - PE = \frac{1}{2}w^T\ {}^sIw - mg\left|{}^sx_c\right|(1-\cos(q)) $$



$$KE = \frac{1}{2} w^T \begin{bmatrix} 0 \\ 0 \\ (m\left|{}^s x_c\right|^2 + I_{czz})\dot{q} \end{bmatrix} = \frac{1}{2}(m\left|{}^s x_c\right|^2 + {}^c I_{zz})\dot{q}^2$$

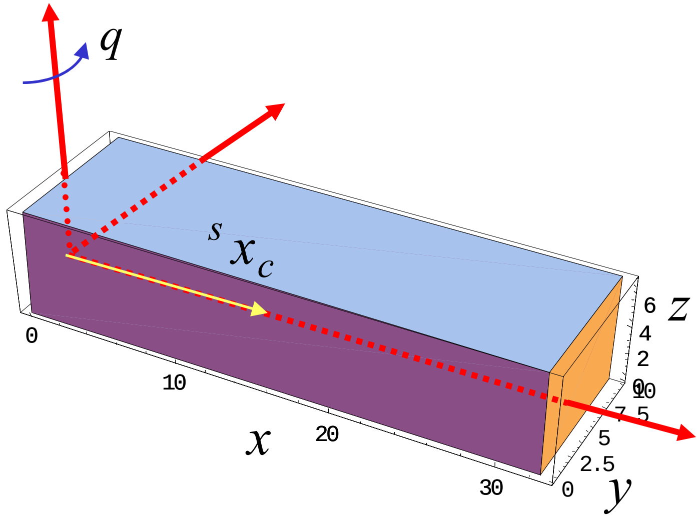
$$PE = mg\left|{}^s x_c\right|(1-\cos(q))$$

$$L = KE - PE = \frac{1}{2}(m\left|{}^s x_c\right|^2 + {}^c I_{zz})\dot{q}^2 - mg\left|{}^s x_c\right|(1-\cos(q))$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{q}}\right) - \frac{dL}{dq} = \frac{d}{dt}\left((m\left|{}^s x_c\right|^2 + {}^c I_{zz})\dot{q}\right) - mg\left|{}^s x_c\right|\sin(q)$$

$$\underline{(m\left|{}^s x_c\right|^2 + I_{czz})\ddot{q} - mg\left|{}^s x_c\right|\sin(q) = 0}$$

Example: KE and dynamics of cuboid model of the upper arm rotating about the shoulder



$$m = 2560\ \text{g} \quad w = \begin{bmatrix} 0 \\ 0 \\ \dot{q} \end{bmatrix} \quad {}^s x_c = \begin{bmatrix} 16.25\cos q \\ 16.25\sin q \\ 0 \end{bmatrix} \text{cm} \quad \dot{x}_s = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$${}^s I = \begin{bmatrix} 30.7 & 0 & 0 \\ 0 & 234.6 & 0 \\ 0 & 0 & 243.9 \end{bmatrix} + 2.560 \begin{bmatrix} 264\sin^2 q & -264\cos q\sin q & 0 \\ -264\cos q\sin q & 264\cos^2 q & 0 \\ 0 & 0 & 264 \end{bmatrix} \text{kg cm}^2$$

$$KE = \frac{1}{2} m \dot{x}_s^T \dot{x}_s + \frac{1}{2} w^T\, {}^s I w + m \dot{x}_s^T (w \times {}^s x_c)$$

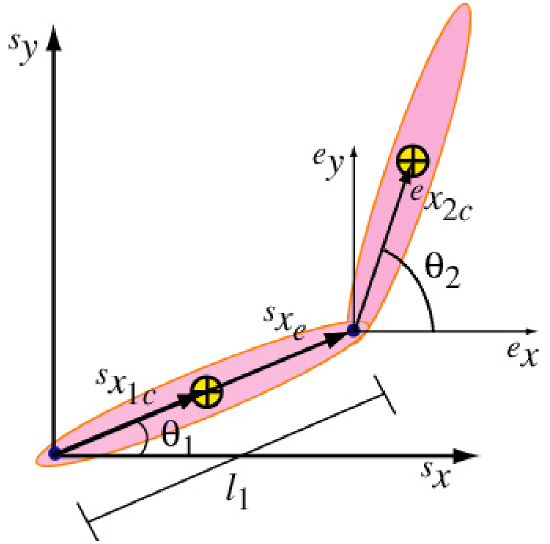
$$KE = \frac{1}{2} w^T\, {}^s I w = 919.74 \dot{q}^2 \ \ \text{kg.cm}^2/\text{sec}^2$$

$$PE = 0$$

$$L = KE - PE = 919.74 \dot{q}^2$$

$$\tau = \frac{d}{dt}\left(\frac{dL}{d\dot{q}}\right) - \frac{dL}{dq} = \frac{d}{dt}(1839.5\dot{q}) - 0 = 1839.5\ddot{q} \ \ \text{g.cm}^2/\text{sec}^2$$

Dynamics of a 2-joint system in the horizontal plane (roll, pitch, yaw angles)



$$ {}^s x_{1c} = \begin{bmatrix} \left|{}^s x_{1c}\right| \cos(\theta_1) \\ \left|{}^s x_{1c}\right| \sin(\theta_1) \\ 0 \end{bmatrix} \quad w_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \quad w_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} \quad {}^c I_1 = \begin{bmatrix} {}^c I_{1xx} & {}^c I_{1xy} & 0 \\ {}^c I_{1yx} & {}^c I_{1yy} & 0 \\ 0 & 0 & {}^c I_{1zz} \end{bmatrix} $$

$$ {}^s I_1 = {}^c I_1 + m_1 \begin{bmatrix} {}^s x_{cy}^2 + {}^s x_{cz}^2 & -{}^s x_{cx}\, {}^s x_{cy} & -{}^s x_{cx}\, {}^s x_{cz} \\ -{}^s x_{cx}\, {}^s x_{cy} & {}^s x_{cx}^2 + {}^s x_{cz}^2 & -{}^s x_{cy}\, {}^s x_{cz} \\ -{}^s x_{cx}\, {}^s x_{cz} & -{}^s x_{cy}\, {}^s x_{cz} & {}^s x_{cx}^2 + {}^s x_{cy}^2 \end{bmatrix} $$

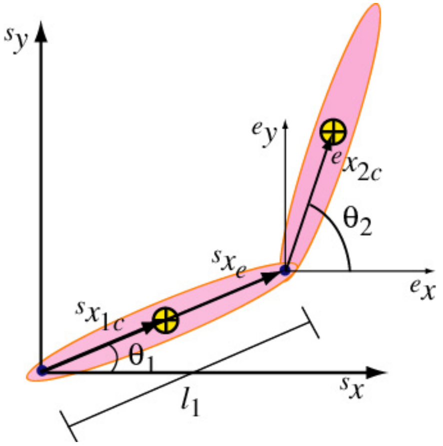
$$ {}^s I_1 = {}^c I_1 + m_1 \begin{bmatrix} \left|{}^s x_{1c}\right|^2 \sin^2(\theta_1) & -\left|{}^s x_{1c}\right|^2 \cos(\theta_1)\sin(\theta_1) & 0 \\ -\left|{}^s x_{1c}\right|^2 \cos(\theta_1)\sin(\theta_1) & \left|{}^s x_{1c}\right|^2 \cos^2(\theta_1) & 0 \\ 0 & 0 & \left|{}^s x_{1c}\right|^2 \end{bmatrix} $$

$$ {}^e I_2 = {}^c I_2 + m_2 \begin{bmatrix} \left|{}^e x_{2c}\right|^2 \sin^2(\theta_2) & -\left|{}^e x_{2c}\right|^2 \cos(\theta_2)\sin(\theta_2) & 0 \\ -\left|{}^e x_{2c}\right|^2 \cos(\theta_2)\sin(\theta_2) & \left|{}^e x_{2c}\right|^2 \cos^2(\theta_2) & 0 \\ 0 & 0 & \left|{}^e x_{2c}\right|^2 \end{bmatrix} $$

$$ KE_1 = \frac{1}{2} w_1^T\, {}^s I_1 w_1 = \frac{1}{2}\left({}^c I_{1zz} + m_1 \left|{}^s x_{1c}\right|^2\right)\dot{\theta}_1^2 $$

$$ KE_2 = \frac{1}{2} m_2\ {}^s\dot{x}_e^T\ {}^s\dot{x}_e + m_2\ {}^s\dot{x}_e^T \left(w_2 \times {}^e x_{2c}\right) + \frac{1}{2} w_2^T\, {}^e I_2 w_2 $$

$$KE_2 = \frac{1}{2} m_2 \; {}^s\dot{x}_e^T \; {}^s\dot{x}_e + m_2 \; {}^s\dot{x}_e^T \left(w_2 \times {}^e x_{2c}\right) + \frac{1}{2} w_2^T \; {}^e I_2 w_2$$



$${}^s\dot{x}_e = w_1 \times {}^s x_e = \begin{bmatrix} -l_1\dot{\theta}_1 \sin(\theta_1) \\ l_1\dot{\theta}_1 \cos(\theta_1) \\ 0 \end{bmatrix} \rightarrow \frac{1}{2} m_2 \; {}^s\dot{x}_e^T \; {}^s\dot{x}_e = \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2$$

$$w_2 \times {}^e x_{2c} = \begin{bmatrix} -\left|{}^e x_{2c}\right| \dot{\theta}_2 \sin(\theta_2) \\ \left|{}^e x_{2c}\right| \dot{\theta}_2 \cos(\theta_2) \\ 0 \end{bmatrix} \rightarrow m_2 \; {}^s\dot{x}_e^T \left(w_2 \times {}^e x_{2c}\right) = m_2 l_1 \left|{}^e x_{2c}\right| \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1)$$

$$\frac{1}{2} w_2^T \; {}^e I_2 w_2 = \frac{1}{2}\left(m_2 \left|{}^e x_{2c}\right|^2 + {}^c I_{2zz}\right)\dot{\theta}_2^2$$

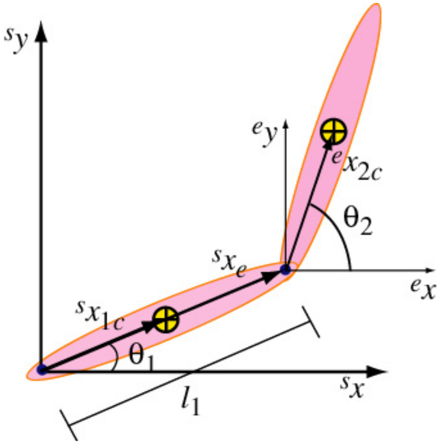
$$KE_2 = \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + m_2 l_1 \left|{}^e x_{2c}\right| \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1) + \frac{1}{2}\left(m_2 \left|{}^e x_{2c}\right|^2 + {}^c I_{2zz}\right)\dot{\theta}_2^2$$

$$KE_1 = \frac{1}{2}\left(m_1 \left|{}^s x_{1c}\right|^2 + {}^c I_{1zz}\right)\dot{\theta}_1^2$$

$$KE = KE_1 + KE_2 = \frac{1}{2}\begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} m_2 l_1^2 + m_1 \left|{}^s x_{1c}\right|^2 + {}^c I_{1zz} & m_2 l_1 \left|{}^e x_{2c}\right| \cos(\theta_2 - \theta_1) \\ m_2 l_1 \left|{}^e x_{2c}\right| \cos(\theta_2 - \theta_1) & m_2 \left|{}^e x_{2c}\right|^2 + {}^c I_{2zz} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$KE = \frac{1}{2} \underline{\dot{\theta}}^T I_{arm} \underline{\dot{\theta}}$$

Inertia of the whole arm: Note that inertia is symmetric, and the physical properties of the system (mass, lengths, etc.) are simple coefficients.



$$KE_1 = \frac{1}{2}\left(m_1\left|{}^{s}x_{1c}\right|^2 + {}^{c}I_{1zz}\right)\dot{\theta}_1^2 = \frac{1}{2}a_1\dot{\theta}_1^2$$

$$KE_2 = \frac{1}{2}m_2 l_1^2\dot{\theta}_1^2 + m_2 l_1\left|{}^{e}x_{2c}\right|\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) + \frac{1}{2}\left(m_2\left|{}^{e}x_{2c}\right|^2 + {}^{c}I_{2zz}\right)\dot{\theta}_2^2$$

$$KE_2 = \frac{1}{2}a_4\dot{\theta}_1^2 + a_3\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) + \frac{1}{2}a_2\dot{\theta}_2^2$$

$$L = KE - PE = \frac{1}{2}a_1\dot{\theta}_1^2 + \frac{1}{2}a_4\dot{\theta}_1^2 + a_3\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) + \frac{1}{2}a_2\dot{\theta}_2^2$$

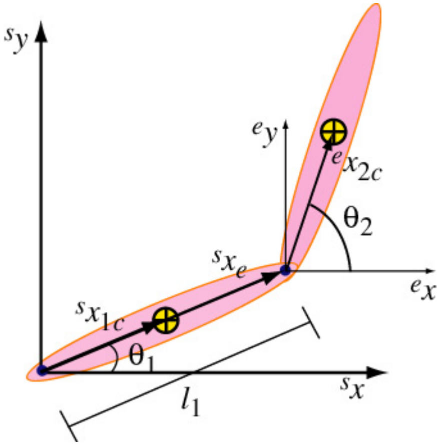
$$\frac{d}{dt}\left(\frac{dL}{d\underline{\dot{\theta}}}\right) - \frac{dL}{d\underline{\theta}} = \underline{\tau}$$

$$\frac{dL}{d\underline{\theta}} = \begin{bmatrix}\dfrac{dL}{d\theta_1}\\[2ex] \dfrac{dL}{d\theta_2}\end{bmatrix} = \begin{bmatrix}a_3\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1)\\ -a_3\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1)\end{bmatrix} \qquad \frac{dL}{d\underline{\dot{\theta}}} = \begin{bmatrix}\dfrac{dL}{d\dot{\theta}_1}\\[2ex] \dfrac{dL}{d\dot{\theta}_2}\end{bmatrix} = \begin{bmatrix}a_1\dot{\theta}_1 + a_4\dot{\theta}_1 + a_3\dot{\theta}_2\cos(\theta_2-\theta_1)\\ a_3\dot{\theta}_1\cos(\theta_2-\theta_1)\end{bmatrix}$$

$$\frac{d}{dt}\left(\frac{dL}{d\underline{\dot{\theta}}}\right) = \begin{bmatrix}a_1\ddot{\theta}_1 + a_4\ddot{\theta}_1 + a_3\ddot{\theta}_2\cos(\theta_2-\theta_1) + a_3\dot{\theta}_2\dfrac{d}{dt}\left(\cos(\theta_2-\theta_1)\right)\\[2ex] a_3\ddot{\theta}_1\cos(\theta_2-\theta_1) + a_3\dot{\theta}_1\dfrac{d}{dt}\left(\cos(\theta_2-\theta_1)\right) + a_2\ddot{\theta}_2\end{bmatrix}$$

$$\frac{d}{dt}\left(\cos(\theta_2-\theta_1)\right) = \frac{d}{d\theta_1}\left(\cos(\theta_2-\theta_1)\right)\dot{\theta}_1 + \frac{d}{d\theta_2}\left(\cos(\theta_2-\theta_1)\right)\dot{\theta}_2 = \dot{\theta}_1\sin(\theta_2-\theta_1) - \dot{\theta}_2\sin(\theta_2-\theta_1)$$

$$\boxed{\underline{\tau} = \begin{bmatrix}\tau_s\\ \tau_e\end{bmatrix} = \begin{bmatrix}a_1\ddot{\theta}_1 + a_4\ddot{\theta}_1 + a_3\ddot{\theta}_2\cos(\theta_2-\theta_1) - a_3\dot{\theta}_2^2\sin(\theta_2-\theta_1)\\ a_2\ddot{\theta}_2 + a_3\ddot{\theta}_1\cos(\theta_2-\theta_1) + a_3\dot{\theta}_1^2\sin(\theta_2-\theta_1)\end{bmatrix}}$$



$$a_1 = m_1\left|{}^s x_{1c}\right|^2 + {}^c I_{1zz}$$

$$a_2 = m_2\left|{}^e x_{2c}\right|^2 + {}^c I_{2zz}$$

$$a_3 = m_2 l_1\left|{}^e x_{2c}\right|$$

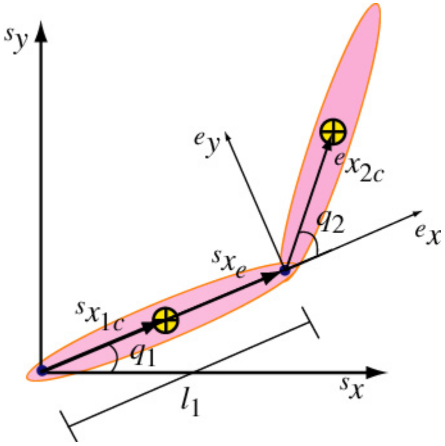
$$a_4 = m_2 l_1^2$$

$$\begin{bmatrix}\tau_s\\ \tau_e\end{bmatrix} = \begin{bmatrix} a_1\ddot{\theta}_1 + a_4\ddot{\theta}_1 + a_3\ddot{\theta}_2\cos(\theta_2-\theta_1) - \overbrace{a_3\dot{\theta}_2^2\sin(\theta_2-\theta_1)}^{\text{Centripetal force}} \\ a_2\ddot{\theta}_2 + a_3\ddot{\theta}_1\cos(\theta_2-\theta_1) + a_3\dot{\theta}_1^2\sin(\theta_2-\theta_1) \end{bmatrix}$$

$$\begin{bmatrix}\tau_s\\ \tau_e\end{bmatrix} = \begin{bmatrix} a_1 + a_4 & a_3\cos(\theta_2-\theta_1) \\ a_3\cos(\theta_2-\theta_1) & a_2 \end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix} 0 & -a_3\dot{\theta}_2\sin(\theta_2-\theta_1) \\ a_3\dot{\theta}_1\sin(\theta_2-\theta_1) & a_2 \end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$$

$$\underline{\tau} = I_{arm}(\underline{\theta})\underline{\ddot{\theta}} + C(\underline{\theta},\underline{\dot{\theta}})\underline{\dot{\theta}}$$

Dynamics of a 2-joint system in the horizontal plane (euler angles)



$$\theta_1 = q_1 \qquad \dot{\theta}_1 = \dot{q}_1$$

$$\theta_2 = q_1 + q_2 \qquad \dot{\theta}_2 = \dot{q}_1 + \dot{q}_2$$

$$KE_2 = \frac{1}{2}\underbrace{m_2 l_1^2}_{a_4}\dot{q}_1^2 + \underbrace{m_2 l_1 \left| {}^e x_{2c} \right|}_{a_3}\dot{q}_1(\dot{q}_1 + \dot{q}_2)\cos(q_2) + \frac{1}{2}\left(\underbrace{m_2 \left| {}^e x_{2c} \right|^2 + {}^c I_{2zz}}_{a_2} \right)(\dot{q}_1 + \dot{q}_2)^2$$

$$KE_2 = \frac{1}{2}a_4\dot{q}_1^2 + a_3\dot{q}_1(\dot{q}_1 + \dot{q}_2)\cos(q_2) + \frac{1}{2}a_2(\dot{q}_1 + \dot{q}_2)^2$$

$$KE_1 = \frac{1}{2}\left(\underbrace{m_1 \left| {}^s x_{1c} \right|^2 + {}^c I_{1zz}}_{a_1} \right)\dot{q}_1^2 = \frac{1}{2}a_1\dot{q}_1^2$$

$$KE = KE_1 + KE_2 = \frac{1}{2}\begin{bmatrix} \dot{q}_1 & \dot{q}_2 \end{bmatrix}\begin{bmatrix} a_1 + a_4 + 2a_3\cos(q_2) + a_2 & a_2 + a_3\cos(q_2) \\ a_2 + a_3\cos(q_2) & a_2 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \frac{1}{2}\underline{\dot{q}}^T \underline{\underline{I}}_{arm}\underline{\dot{q}}$$

$$L = KE - PE = KE - 0$$

$$\underline{\tau} = \begin{bmatrix} \tau_s \\ \tau_e \end{bmatrix} = \frac{d}{dt}\left(\frac{dL}{d\underline{\dot{q}}} \right) - \frac{dL}{d\underline{q}}$$

$$\begin{bmatrix} \tau_s \\ \tau_e \end{bmatrix} = \begin{bmatrix} a_1 + a_4 + 2a_3\cos(q_2) + a_2 & a_2 + a_3\cos(q_2) \\ a_2 + a_3\cos(q_2) & a_2 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -a_3\dot{q}_2\sin(q_2) & -a_3(\dot{q}_1 + \dot{q}_2)\sin(q_2) \\ a_3\sin(q_2)\dot{q}_1 & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

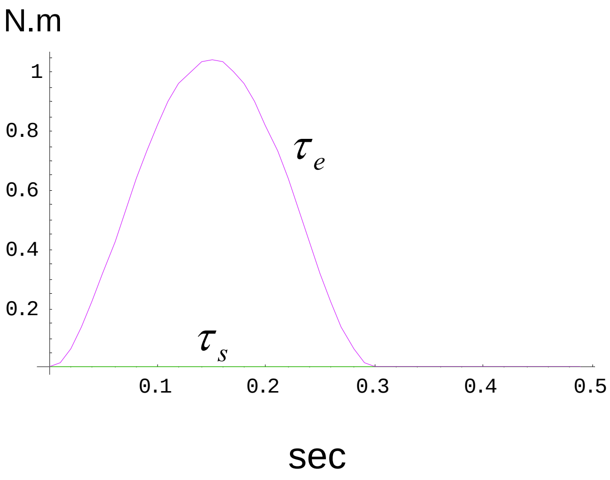
$$\underline{\tau} = I_{arm}(\underline{q})\underline{\ddot{q}} + C(\underline{q},\underline{\dot{q}})\underline{\dot{q}}$$

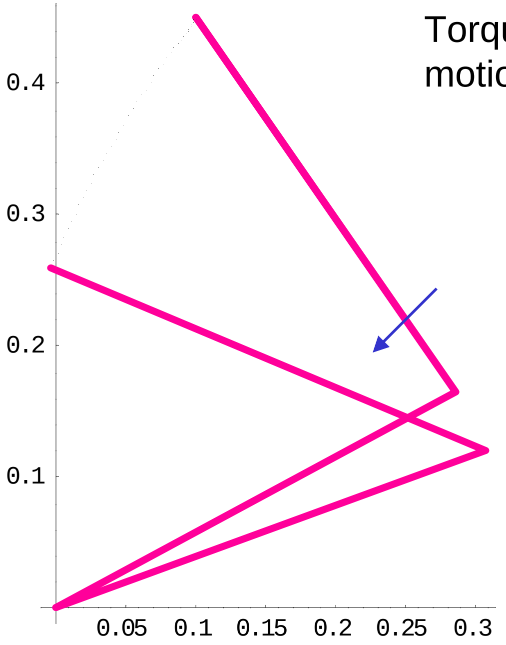
$$\begin{bmatrix}\tau_s \\ \tau_e\end{bmatrix} = I_{arm}\begin{bmatrix}\ddot{q}_1 \\ \ddot{q}_2\end{bmatrix} + \begin{bmatrix} \overbrace{-2a_3\dot{q}_1\dot{q}_2\sin(q_2)}^{\text{Coriolis force}} \overbrace{-a_3\dot{q}_2^2\sin(q_2)}^{\text{Centripetal force}} \\ a_3\dot{q}_1^2\sin(q_2)\end{bmatrix}$$

$$\underline{\ddot{q}} = I_{arm}^{-1}\left(\underline{\tau} - C(\underline{q},\underline{\dot{q}})\underline{\underline{\dot{q}}}\right)$$

Example: Effect of interaction torques

Command a flexion torque on the elbow

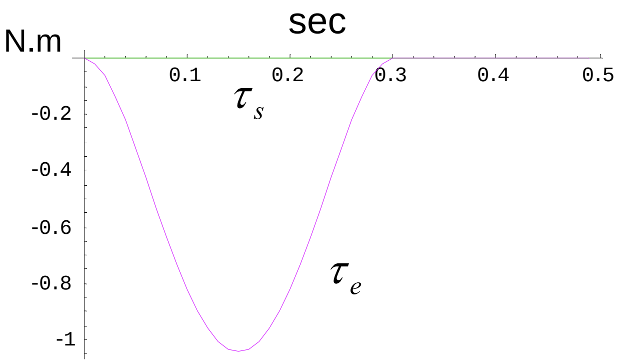


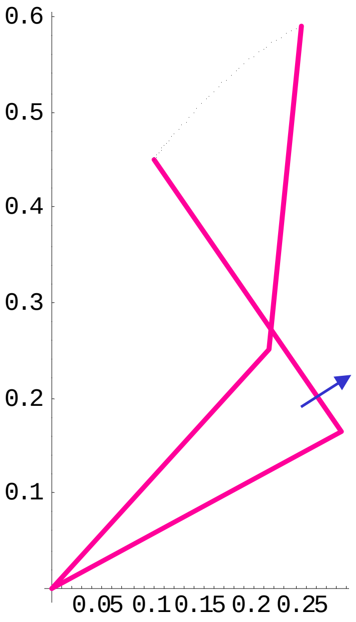


Torque on the elbow causes motion of the elbow and shoulder

Example: Effect of interaction torques

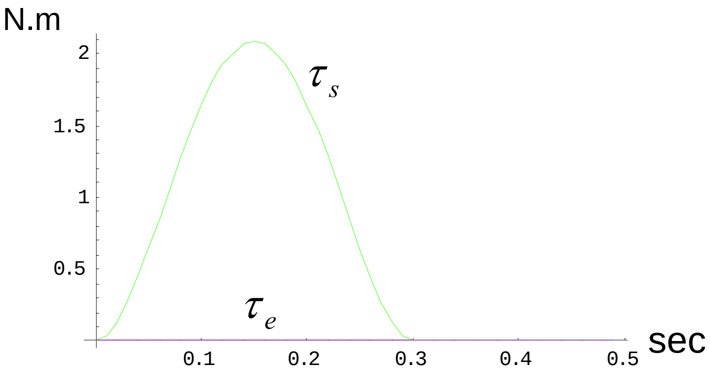
Command an extension torque on the elbow

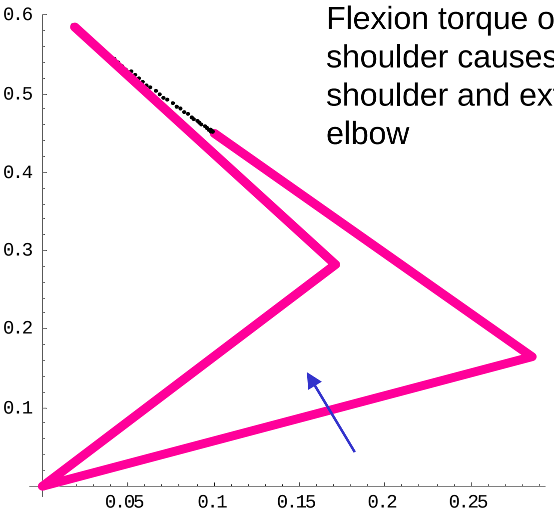




Extension torque on the elbow causes extension of elbow and flexion of the shoulder

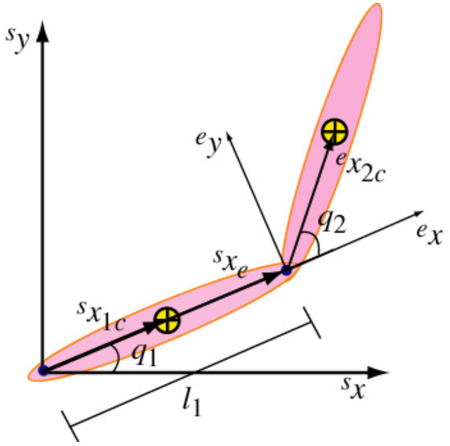
Command a flexion torque on shoulder





Flexion torque on the shoulder causes flexion of shoulder and extension of elbow

Physical interpretation of the equations



$$a_1 = m_1 \left| {}^s x_{1c} \right|^2 + {}^c I_{1zz} \qquad a_2 = m_2 \left| {}^e x_{2c} \right|^2 + {}^c I_{2zz}$$

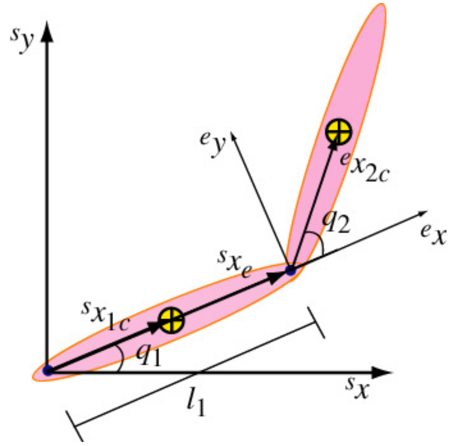
$$a_3 = m_2 l_1 \left| {}^e x_{2c} \right| \qquad a_4 = m_2 l_1^2$$

$$\begin{bmatrix} \tau_s \\ \tau_e \end{bmatrix} = \begin{bmatrix} a_1 + a_4 + 2a_3 \cos(q_2) + a_2 & a_2 + a_3 \cos(q_2) \\ a_2 + a_3 \cos(q_2) & a_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}$$
$$+ \begin{bmatrix} -a_3 \dot{q}_2 \sin(q_2) & -a_3(\dot{q}_1 + \dot{q}_2)\sin(q_2) \\ a_3 \sin(q_2)\dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

1. Immobilize the second link.

$$\dot{q}_2 = 0 \qquad \ddot{q}_2 = 0$$
$$\tau_s = \underbrace{\left(a_1 + a_4 + 2a_3 \cos(q_2) + a_2\right)}\ddot{q}_1 + \cdots$$

Inertia seen by the first joint when the second joint is immobilized.
Reaches maximum value when the arm is open.



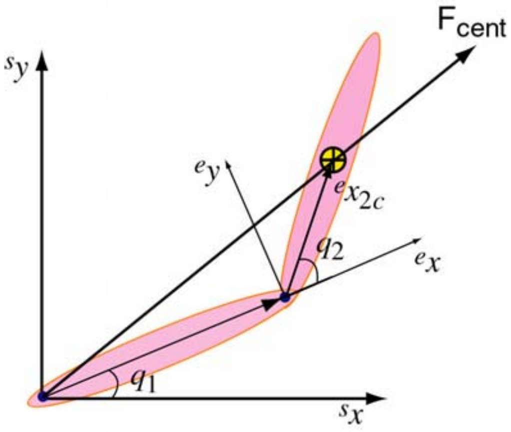
2. Immobilize the first link.

$$\dot{q}_1 = 0 \qquad \ddot{q}_1 = 0$$

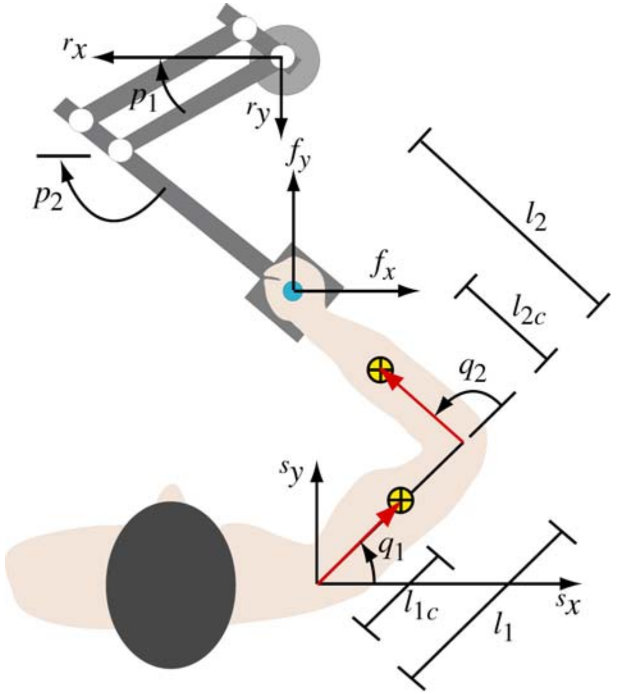
$$\tau_s = \underbrace{\left(a_2 + a_3 \cos(q_2)\right)}_{\text{Reaction forces acting on the first link when the second link accelerates.}}\ddot{q}_2 + \cdots$$

3. Centripetal forces acting on link 2 due to rotation of link 1

$$\tau_e = \cdots + a_3 \sin(q_2)\dot{q}_1^{\;2}$$



Estimating inertial parameters of the human arm



$$I_r(p)\ddot{p}+C_r(p,\dot{p})\dot{p}=\tau_r-J_r^T(p)F_{hand}$$

$$I_h(q)\ddot{q}+C_h(q,\dot{q})\dot{q}=\tau_h+J_h^T(q)F_{hand}$$

$$\tau_h \equiv \begin{bmatrix}\tau_{hs}\\ \tau_{he}\end{bmatrix}=\begin{bmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix}(q-q_d)+\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}(\dot{q}-\dot{q}_d)$$

q_d : joint angles at the start of perturbations

$$\dot{q}_d=0$$

$$a_1=m_1 l_{1c}^2+{}^c I_{1zz} \qquad a_2=m_2 l_{2c}^2+{}^c I_{2zz}$$

$$a_3=m_2 l_1 l_{2c} \qquad a_4=m_2 l_1^2$$

$$I_h=\begin{bmatrix}a_1+a_4+2a_3\cos(q_2)+a_2 & a_2+a_3\cos(q_2)\\ a_2+a_3\cos(q_2) & a_2\end{bmatrix}$$

$$C_h=\begin{bmatrix}-a_3\dot{q}_2\sin(q_2) & -a_3(\dot{q}_1+\dot{q}_2)\sin(q_2)\\ a_3\sin(q_2)\dot{q}_1 & 0\end{bmatrix}$$

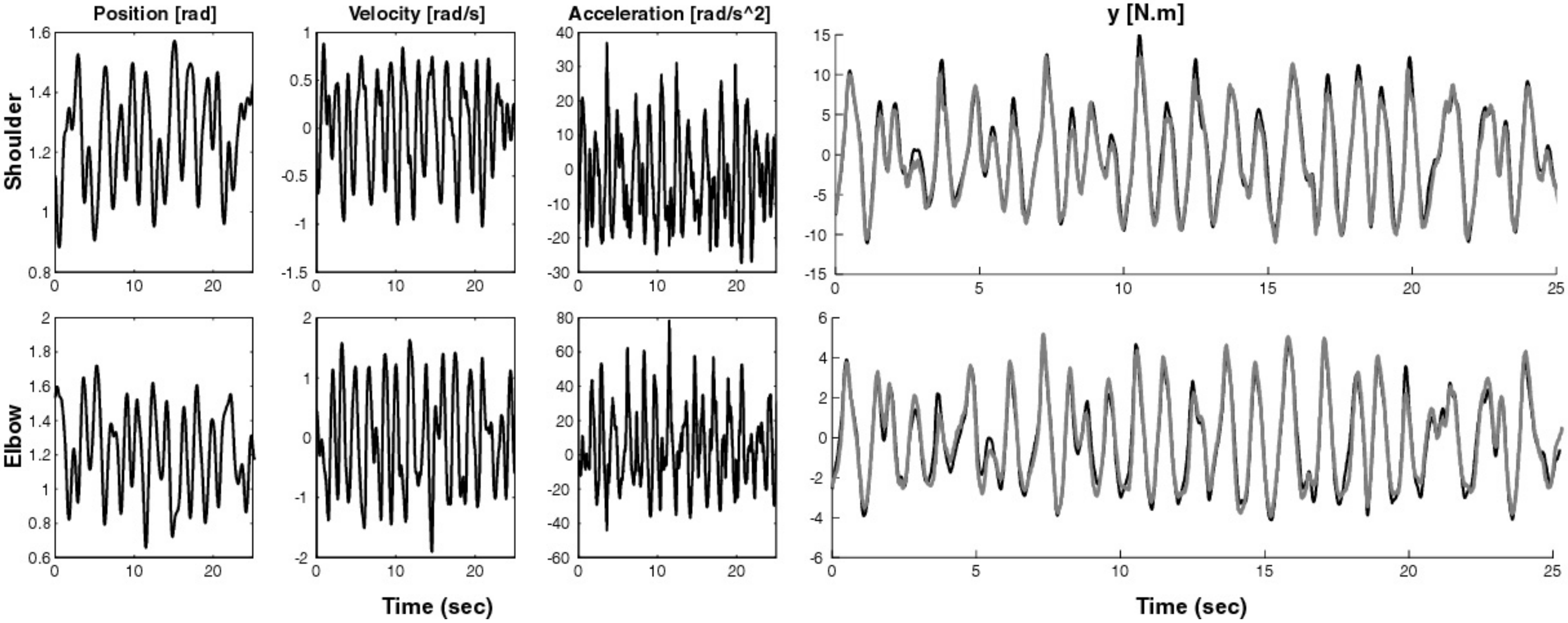
$$J_h^T(p)F_{hand}\equiv\begin{bmatrix}y_1\\ y_2\end{bmatrix}$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}\left(a_1+a_4+2a_3\cos(q_2)+a_2\right)\ddot{q}_1+\left(a_2+a_3\cos(q_2)\right)\ddot{q}_2-a_3\dot{q}_1\dot{q}_2\sin(q_2)-a_3(\dot{q}_1+\dot{q}_2)\dot{q}_2\sin(q_2)\\ a_2+a_3\cos(q_2)\ddot{q}_1+a_2\ddot{q}_2+a_3\sin(q_2)\dot{q}_1^2\end{bmatrix}-\tau_h$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \ddot{q}_1 & \ddot{q}_2 & \ddot{q}_2\cos(q_2) - \dot{q}_1\dot{q}_2\sin(q_2) - (\dot{q}_1+\dot{q}_2)\dot{q}_2\sin(q_2) & \Delta q_1 & \Delta q_2 & 0 & 0 & \dot{q}_1 & \dot{q}_2 & 0 & 0 \\ 0 & 1 & \cos(q_2)\ddot{q}_1 + \sin(q_2)\dot{q}_1^2 & 0 & 0 & \Delta q_1 & \Delta q_2 & 0 & 0 & \dot{q}_1 & \dot{q}_2 \end{bmatrix} \begin{bmatrix} a_1 + a_4 \\ a_2 \\ a_3 \\ k_{11} \\ k_{12} \\ k_{21} \\ k_{22} \\ b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{bmatrix}$$

$$\underline{y}_1 \equiv \begin{bmatrix} y_{1(t=0)} \\ \vdots \\ y_{1(t=25)} \end{bmatrix} = G_1 \underline{a} \quad \rightarrow \underline{a} = G_1^{-1} \underline{y}_1$$

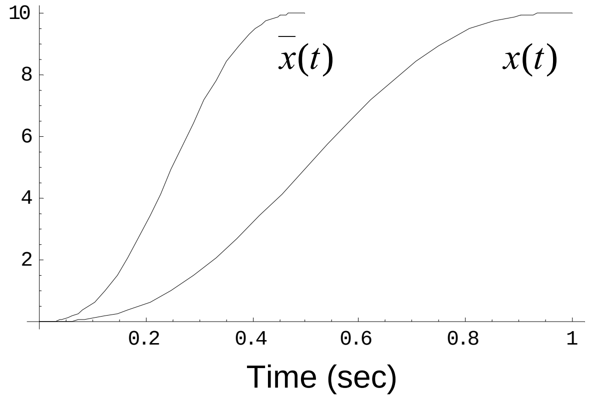
$$a_1 + a_4 = 0.429 \quad a_2 = 0.143 \quad a_3 = 0.132 \quad \text{kg.m}^2$$



Moving at different speeds: how torques change

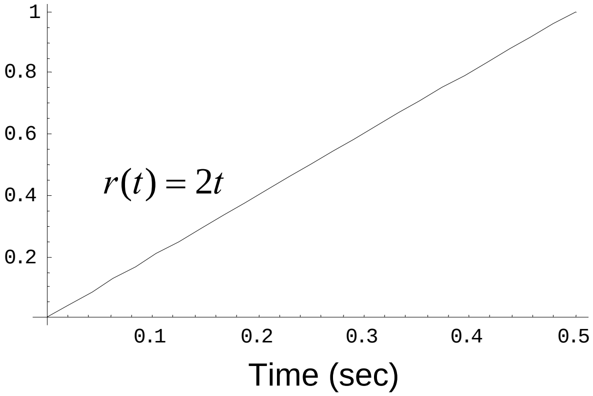
Objective: Is there a simple relation between torques when the speed of movements scales up?

Example: Imagine that one makes a movement from point 1 to 2 in 1 second. How much more joint torque would be needed to make the same movement in 0.5 seconds?

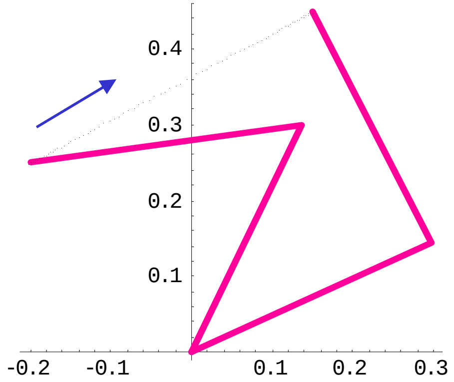


$$\bar{x}(t) = x(r(t))$$

$$r(t) = 2t$$



Moving at different speeds: how torques change



original trajectory: $q(t)$ sped up trajectory: $\bar{q}(t) = q(r(t))$

$$\frac{d\bar{q}(t)}{dt} = \frac{dq(r)}{dt} = \frac{dq(r)}{dr}\dot{r} = q'(r)\dot{r}$$

$$\frac{d^2\bar{q}(t)}{dt^2} = \frac{d\left(q'(r)\dot{r}\right)}{dt} = \frac{dq'(r)}{dt}\dot{r} + q'(t)\ddot{r} = \frac{dq'(r)}{dr}\dot{r}^2 + q'(t)\ddot{r} = q''(r)\dot{r}^2 + q'(t)\ddot{r}$$

for trajectory: $q(t) \rightarrow \tau(t) = I(q)\ddot{q} + \dot{q}C(q)\dot{q}$

sped up trajectory: $\bar{q}(t) \rightarrow \bar{\tau}(t) = I(\bar{q})\dfrac{d^2\bar{q}(t)}{dt^2} + \dfrac{d\bar{q}(t)}{dt}C(\bar{q})\dfrac{d\bar{q}(t)}{dt}$

$$\bar{\tau}(t) = I(q(r))\left(q''(r)\dot{r}^2 + q'(t)\ddot{r}\right) + q'(r)\dot{r}C(q(r))q'(r)\dot{r}$$

$$\bar{\tau}(t) = \left[I(q(r))q''(r) + q'(r)C(q(r))q'(r)\right]\dot{r}^2 + I(q(r))q'(t)\ddot{r}$$

$$\bar{\tau}(t) = \tau(r)\dot{r}^2 + I(q(r))q'(r)\ddot{r}$$

Simple scaling of time: $r = 2t \quad \dot{r} = 2 \quad \ddot{r} = 0$

$$\bar{\tau}(t) = 4\tau(2t)$$

To move twice as fast, you'd need four times as much torque to compensate for inertial dynamics